Physics 435 : Grading

11 Home Work: 30%

Discussion Section: 5% attendance

3 Midterms : 30%

Final: 35%

I have heavily weighted the homework since it's the only way of learning E&M at this level.

Jim Wiss Office Hours Thursdays 6:30pm to 8:00 pm Room to be announced by email appointment. (jew@illinois.edu)

Mid Terms (in class):

Fri Feb. 22
Fri Mar. 15
Fri Apr. 26

Home Work due Friday in HW box

Friday < 5pm => 100% credit
Monday < 5pm => 90% credit
Thursday < 11 pm => 70% credit

I hope lecture notes will be available at

T.I.S Bookstore
707 S. 6th St.
Champaign

http:// courses.physics.illinois.edu/phys435/sp2013/

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Welcome to Physics 435!

You saw 3 of 4 Maxwell Eqn in Physics 212

Gauss's Law : $\varepsilon_0 \int \vec{E} \cdot d\vec{a} = Q_{encl}$

Faraday's Law: $\varepsilon = \oint \vec{E} \bullet d\vec{\ell} = \frac{d}{dt} \int \vec{B} \cdot d\vec{a}$

Ampere-Maxwell $\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \varepsilon_0 \frac{d}{dt} \left[\int \vec{E} \cdot d\vec{a} \right] \right)$

So why study E&M further?

Great applications of crucial mathematics:

a) Vector Calculus
b) Orthogonal functions
c) Curvilinear coordinate systems

Physics beyond Phys 212

a) Differential form of Maxwell Eq,
b) E&M in materials : Bound charge & currents
c) Vector Potentials in Mech and Quant Mech
d) Wave guides
e) Radiation theory
f) Relativistic transformations of E&M fields

E&M is the most practical branch of classical physics

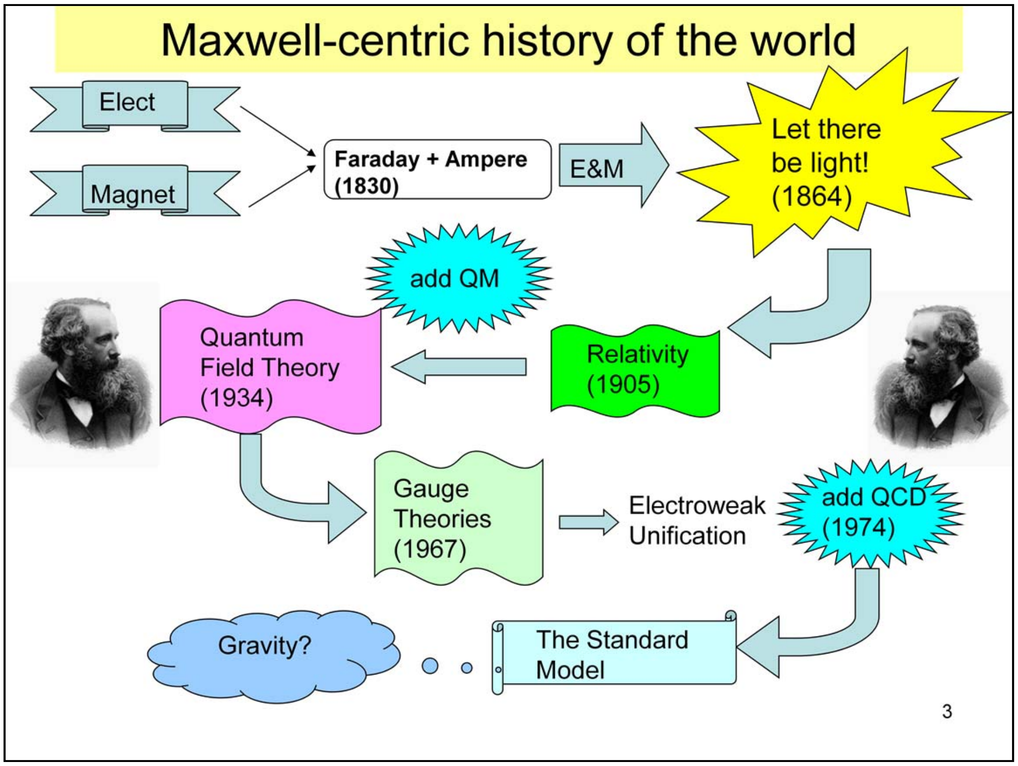
It is correct as originally written by Maxwell in 1864:

a) Relativistically
b) Quantum mechanically

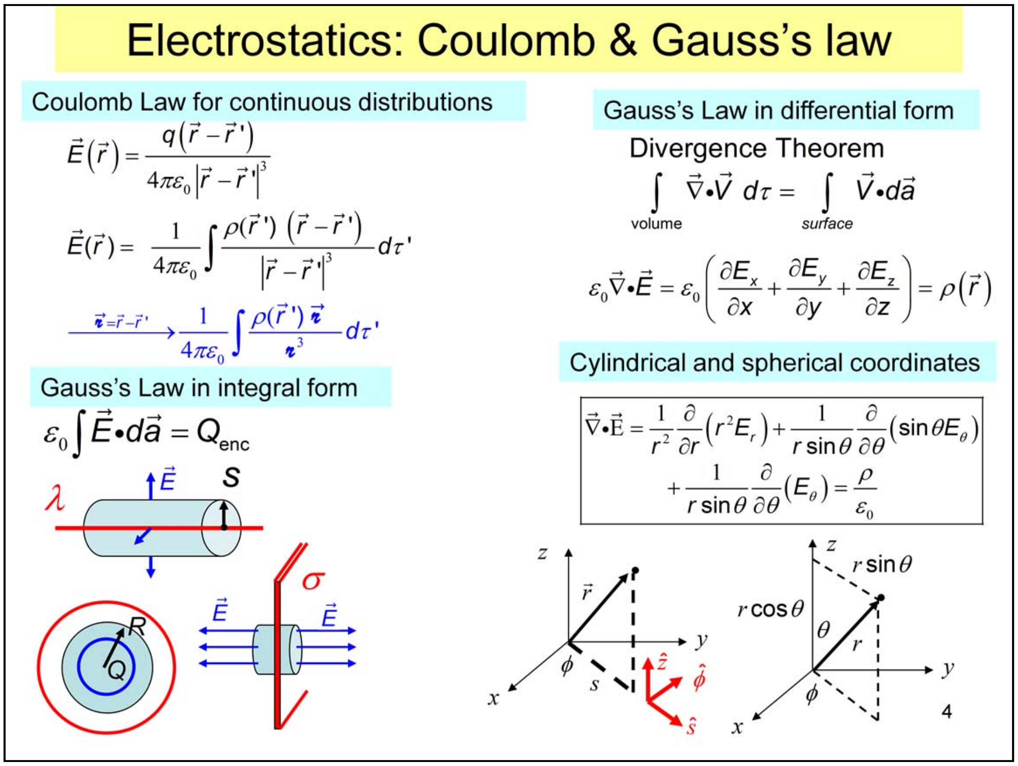
E&M forms theoretical template for the successful gauge theory strong & weak interactions or the Standard Model.

Most of you have already been exposed to the basic physics of E&M by taking Physics 212 or its equivalent. Last semester we taught three of the four Maxwell Equations in integral form. You might recognize them. One of the important motivations for a second, two-semester course in E&M is that it gives us the excuse to introduce some really important and interesting mathematics that you will use in subsequent physics courses such as quantum mechanics. We will make extensive use of vector calculus such as divergence and curls. We will discuss the role of orthogonal functions in the solutions of partial differential equations. Fourier expansions is a possibly familiar use of orthogonal functions. Finally we will discuss the use of curvilinear coordinate systems such as spherical and cylindrical coordinate systems. These are generally the most elegant ways of handling spherical or cylindrically symmetric systems. The first chapter of your textbook , Electrodynamics by David J. Griffiths gives a very insightful overview of much of this mathematics. Of course we will also introduce some important physics beyond Physics 212. This includes obtaining the differential forms of Maxwell's Eqn which nicely complement the integral forms that you are familiar with. The integral forms are particularly useful in very quickly solving highly symmetric problems such as the E-fields from spherically symmetric charge distributions. The differential forms are particularly useful in developing the theory of electrodynamics and in solving electromagnetic wave problems. We also discuss some of the very interesting and elegant physics involving electromagnetic fields in materials. We will show that the field is much richer than just substituting epsilon for epsilon_0 and mu for mu_0. The more complete theory makes extensive use of the concepts of bound charges and bound currents. I learned a lot about bound charges and currents while preparing to teach this course. We will also discuss the use of magnetic potentials. The most interesting magnetic potential is actual a vector potential with three components. The magnetic field can be written as derivatives of the vector potential much like the electric field can be written as the derivative of the scalar potential or voltage. It is fair to say that much of the modern physics created after 1960 that describes the subatomic world was inspired by the use of the vector potential to inject magnetic forces into Lagrangian dynamics and quantum mechanics. We will also discuss the radiation of electromagnetic waves, the theory of wave guides, and the role of relativity in understanding electrodynamics. The theory of relativity was invented by Einstein in the early1900's who was exploring basic symmetry of electrodynamics.

Not only is E&M the most practical branch of classical physics but it is the only classical physics which is correct as far as we know. It requires no relativistic modifications although relativity gives considerable insight into the relationship between E and B fields. It carries over to quantum mechanics rather seamlessly as well and formed the basic template for modern field theories. Historically a good rule of thumb seems to be that the closer a new theory is to E&M the more likely the new theory is true.



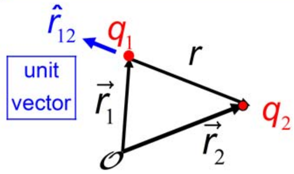
E&M has had a long and distinguished history dating from the early 1800. By 1830 or so it became clear that electricity and magnetism were unified. Faraday discovered a changing magnetic field could create an electric field. Ampere, Oersted , Henry and others showed electric currents could create magnetic fields. Our hero (James Clerk Maxwell) comes on the scene with a crucial paper in 1864 which concludes that just like Faraday's discovery that a changing B field creates an E field, a changing E-field creates a B-field. Maxwell added a new term – the displacement current– to Ampere's Law. This new term was partially motivated by the observation that Ampere's law was inconsistent with charge conservation. It restores a basic symmetry between electricity and magnetism and most importantly allows for the propagation of electromagnetic waves in a vacuum or in the absence of charge or currents. I think of electromagnetic waves as a self generating dynamo where the changing B field creates a changing E field which in turn creates a changing B field. Maxwell gave the first correct picture for the nature of light and "man-made" light or radio waves. Maxwell was able to correctly predict the speed of light based on lab bench constants such as epsilon_0 and mu_0– although he couldn't believe that the speed of light was independent of reference frame even though this was a consequence of Maxwell's Eq. Precise measurements of the speed of light which used the high speed of the earth's surface relative to the fixed stars failed to find any dependence on relative velocity. Einstein realized that the only way the velocity of light could be frame independent is if time intervals depend on the frame. This was the death of the universal clock of Newtonian Physics. The bad news that follows from frame dependant time is that all of the Physics 211 mechanics you learned in high school and in Physics 211 is only approximate! The good news is all of the EM you learned in Physics 212 is exactly right! In an attempt to make electromagnetism consistent with quantum mechanics Heiter and others developed Quantum field theory in the early 30's. By the early 1960's theorists were realizing that E&M probably formed a beautiful template for subatomic interactions such as the weak force which plays a crucial role in beta decay and neutrino interactions. Beta decay, for example, explains why nuclei have nearly equal numbers of neutrons and protons. A key ingredient in these E&M inspired theories is gauge invariance which concerns the astonishing amount of freedom that one has in defining the vector potential. By 1967 a (nearly) successful unification of E&M and the Weak Interaction was proposed by Weinberg and Salam – a small but important inconsistency in the WS model was resolved in 1974 with the discovery of the charmed quark. In 1974 the highly successful electro-weak model of Weinberg and Salam served as a template for QCD which essentially unified E&M , the weak, and the strong interaction (the other subatomic force) into a theory called "the Standard Model". The only force not described by the Standard model to date is gravity – paradoxically the first classical force with a reasonable model by Newton!



Here is what we will discuss in this first set of lectures. Some of this material may be familiar from Physics 212. We will discuss computing the E-field from Coulomb's law both for point charges and continuous distributions. We will introduce Griffiths' script r notation for the relative position between source and observer. This will be followed by a discussion of Gauss's law and some of the classic Gauss's law geometries. We next write Gauss's law in differential form which is better for some application. We will typically work in Cartesian as well as curvilinear coordinate systems such as spherical and cylindrical coordinates.

Coulomb's Law

$\vec{F}_{12}$ ="force on 1 due to 2"



Coulomb's Law

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r}_{12};\quad r = \left|\vec{r}_1 - \vec{r}_2\right|$$

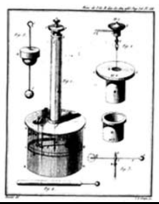
$$\frac{1}{4\pi\varepsilon_0} \approx 9\times 10^9 \frac{N\bullet m^2}{C^2} \quad \varepsilon_0 = 8.85\times 10^{-12}\frac{C^2}{N\bullet m^2}$$

We can also write this as

$$\vec{F}_{12} = \frac{q_1 q_2\left(\vec{r}_1 - \vec{r}_2\right)}{4\pi\varepsilon_0\left|\vec{r}_1 - \vec{r}_2\right|^3} \text{ where}$$

$$\vec{r}_1 - \vec{r}_2 = \left(x_1 - x_2 \quad y_1 - y_2 \quad z_1 - z_2\right)$$

$$\left|\vec{r}_1 - \vec{r}_2\right| = \sqrt{\left(x_1 - x_2\right) + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2}$$



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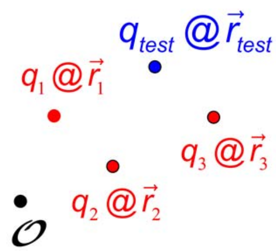
Coulomb’s law is at the center of electrostatics and we will obtain many of the key electrostatic results using this formulation. We will treat it as an experimental fact. Coulomb says the force between two forces is proportional to the product of their charges and inversely proportional to the square of the distance between them. Coulomb’s constant will typically be written as 1/(4 pi epsilon_0) where epsilon_0 is 8.85 times 10^-12 in the MKS system which we will use throughout Physics 435 and 436. The force units are Newtons, the charge units are Coulomb and distance is measured in meters. 1/(4 pi epsilon_0) implies an enormous force between objects holding a Coulomb of charge – primarily since this is an astronomical amount of charge. The charge of an electron or proton is 1.6 times 10^{-19} Coulombs. The electrostatic force is a central force which means it points along the line joining the two charges. We can thus write the force components between two charges located with Cartesian coordates (x1,y1,z1) and (x2,y2,z2) in terms of the indicated expression based on the difference of coordinates or in terms of the vector difference between the displacement vectors r1 and r2. The form (r1-r2)/|r1 – r2|^3 can be viewed as the unit vector r-hat = (r1-r2)/|r1 – r2| which sets the direction of the force times an inverse r squared force or 1/|r1 – r2|^2. This is where the cubes are coming from.

Electrical fields and superposition



$$\vec{F}_{test} = \sum_{j=1,2,3} \frac{q_{test} q_j \left(\vec{r}_{test} - \vec{r}_j\right)}{4\pi\varepsilon_0 \left|\vec{r}_{test} - \vec{r}_j\right|^3} \text{ or } \vec{F}_{test} = \sum_{j \neq test} \frac{q_{test} q_j \left(\vec{r}_{test} - \vec{r}_i\right)}{4\pi\varepsilon_0 \left|\vec{r}_{test} - \vec{r}_j\right|^3}$$

No self force!

$$\vec{E}(\vec{r}_{test}) = \frac{\vec{F}_{test}}{q_{test}} = \sum_{j \neq test} \frac{q_j \left(\vec{r}_{test} - \vec{r}_i\right)}{4\pi\varepsilon_0 \left|\vec{r}_{test} - \vec{r}_j\right|^3}$$

$\boxed{\text{In continuous limit:}}$ $\vec{r}_{test} \rightarrow \vec{r};\ \sum \rightarrow \int \quad ;\ q_j \rightarrow \rho\left(\vec{r}\,'\right) d\tau$

$$\vec{E}(\vec{r}\,) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')\,\hat{r}}{\left|\vec{r} - \vec{r}\,'\right|^3} d\tau' \xrightarrow[\hat{r} = \vec{\mathscr{r}}/\mathscr{r}]{\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'} \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')\,\vec{\mathscr{r}}}{\mathscr{r}^3} d\tau'$$

Here $\int [\ ]\, d\tau' = \iiint [\ ]\, dx'dy'dz'$ and $\hat{r} = \dfrac{\left(\vec{r} - \vec{r}\,'\right)}{\left|\vec{r} - \vec{r}\,'\right|} = \dfrac{\vec{\mathscr{r}}}{\left|\vec{\mathscr{r}}\right|}$

$\vec{E}(\vec{r}\,)$ is a vector field. A function with 3 components that depends on position. $\vec{F}(\vec{r}\,) = q\vec{E}(\vec{r}\,)$. Think of q as test charge. $\vec{E}(\vec{r}\,)$ due to all other charges.

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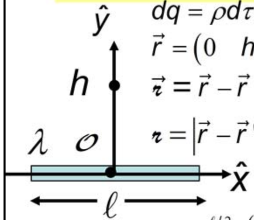
One critical idea in Electricity and Magnetism is the principle of superposition. If you have 3 charges and want to find the force on a 4th "test" charge, you add the force vectors for the force between test and the other charges q1,q2,q3. You do not include the force of qtest on itself which would be infinity anyway. There are no self-forces or fields. The force on the test charge can be used to measure the E-field at the position rtest. We take the force on the test charge and divide by the test charge's charge to find the electric field at that point. We frequently will want the electrical field due to a continuous distribution of charges. Essentially we convert the sum over other charges to an integral. The key concept here is the charge density rho or the number of Coulombs per infinitesimal volume which we will write as d tau throughout this course. The field is evaluated at a particular displacement vector called r. I find it convenient to call the point where we want to know the the E-field the "observation" position r. The other vector , r', is the "source" vector or the displacement of each charge element that is being integrated over in order to compute the field at the observation point. Because the integral is over "source" point coordinates, I write the volume element as d tau'. The vector between the source and observer shows up so often in EM that Griffiths invented a special "script" symbol for the relative displacement r – r' which we will use throughout Physics 435 and 436. The Griffiths r with a vector hat is the relative displacement vector while without the vector hat stands for the magnitude of the Griffiths r. In general we will need a triple integral and in Cartesian coordinated d tau' = dx' dy' dz'. Of course, sometimes a three dimensional integral is not necessary. For example the charges might be confined to a plane or other surface. Then the distribution would be categorized by a surface density sigma. Sometimes the charges lie on line or a curved path in which case we use a single integral and a linear charge density Lambda.

Continuous Distribution Example



$dq = \rho d\tau' \to \lambda\, dx'$

$\vec{r} = \begin{pmatrix}0 & h\end{pmatrix};\ \vec{r}' = \begin{pmatrix}x' & 0\end{pmatrix}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = \begin{pmatrix}-x' & h\end{pmatrix}$

$\mathscr{r} = |\vec{r} - \vec{r}'| = \sqrt{x'^2 + h^2}$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\vec{\mathscr{r}}}{\mathscr{r}^3}\, d\tau'}$$

$$\begin{pmatrix}E_x & E_y\end{pmatrix} = \frac{1}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2} \frac{\begin{pmatrix}-x' & h\end{pmatrix}}{\sqrt{x'^2+h^2}^3}\lambda\, dx'$$

$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2} \frac{-x'}{\sqrt{x'^2+h^2}^3}\, dx' = 0 \text{ (odd integrand)}$$

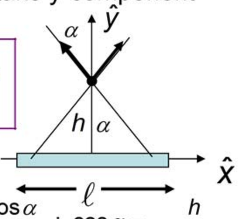
$$E_y = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2} \frac{h}{\sqrt{x'^2+h^2}^3}\, dx'$$

$$\int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} \text{ (integral table)}$$

$$E_y = \frac{h\lambda}{4\pi\varepsilon_0}\left(\frac{x}{h^2\sqrt{x^2+h^2}}\right)_{-\ell/2}^{\ell/2} = \frac{h\lambda}{2\pi\varepsilon_0}\left(\frac{\ell/2}{h^2\sqrt{\left(\ell/2\right)^2+h^2}}\right)$$

$$\to \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 h}\left(\frac{\ell}{\sqrt{\ell^2+4h^2}}\right)\hat{y}$$

Informal Method Add E in ±x pairs to cancel E_x and take y-component



$$dE_y = \frac{\lambda dx\ \cos\alpha}{4\pi\varepsilon_0\left(x^2+h^2\right)};\ \cos\alpha = \frac{h}{\sqrt{x'^2+h^2}}$$

$$E_y = 2 \otimes \frac{\lambda h}{4\pi\varepsilon_0}\int_0^{\ell/2} \frac{dx'}{\left(x'^2+h^2\right)^{3/2}}$$

Limits of $\boxed{E_y = \frac{\lambda}{2\pi\varepsilon_0 h}\left(\frac{\ell}{\sqrt{\ell^2+4h^2}}\right)}$:

As $\ell \to \infty$ $\frac{\ell}{\sqrt{\ell^2+4h^2}} \to 1$

and $E_y \to \frac{\lambda}{2\pi\varepsilon_0 h}$ (line charge)

As $\ell \to 0$ $\frac{\ell}{\sqrt{\ell^2+4h^2}} \to \frac{\ell}{2h}$ and

$E_y \to \frac{\lambda\ell}{4\pi\varepsilon_0 h^2} = \frac{q}{4\pi\varepsilon_0 h^2}$ (point charge)

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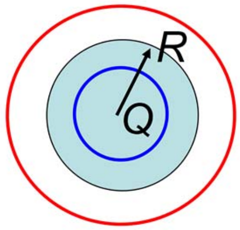
Here is an example of computing the field due to a finite, uniform line of charges. In order to simplify the problem, we put the observation point a distance Z above the center of the line of charge. The advantage of “centering” the observation point is that the component of the E-field parallel to the line of charge (i.e. the x-component) will cancel from +x and –x pairs leaving just the z-component. Our first step is to write an expression for the charge element dq. Since this is a 1 dimensional distribution the density expression rho d tau’ is replaced by lambda dx’ where lambda is in Coulombs/meter. We next write the observation point and source point vectors referenced to our origin (O) and coordinate system. We write just two components for simplicity and since nothing is happening in z. The observation point is at x=0 y=h. The source points are at x = x’ and y = 0. We take the difference of these two vectors to get r-r’ and use sqrt of sum of squares of components to construct |r – r’| . We next transcribe our expression for E from rho d tau’ to lambda dx’ The limits of integration are from -L/2 to L/2 since that is the only region with charge. We write the E and r-r’ vectors with explicit components. The Ex integral has an odd integrand over a symmetric domain and thus vanishes. For the Ey integral you will probably need an integral table. Fortunately a definite integral for this form exists. We use the integral table form with “a” =h and x = x’ and subtract the lower limit from the upper limit to get our final Ey expression. I hope you will find the forgoing straightforward although a bit tedious. We can cut through much of the tedium by an informal approach. We will integrate from 0 to L/2 and double our answer for Ey and realize that each +x charge will cancel the Ex field from each –x charge. We write the E- field element as the charge elements (Lambda dx) divided by the square of the observer to source distance which is the square of the hypotenuse of either right triangles times the constant. The E-field lies along the line from the source to the observation point and we want to take the Ey component by taking the cosine of angle with respect to the E-field direction and the y-axis which is marked alpha. This cosine is just the adjacent side (h) divided by the hypotenuse [sqrt(h^2+x^2)]. The sqrt(h^2+x^2) times r^2 gives us the (h^2+x^2)^(3/2) which appears in the denominator. We integrate from 0 to L/2 and double the result. We are left with the same integral as that in the formal treatment to the left. Frequently on exams or homework you will want to check your result by going to various limits where you know the answer. One limit is where the length of the charged region approaches infinity leaving us with an infinite line of charge. In that limit L/sqrt(L^2+4h^2) approaches 1 and we are left with the “line charge” expression that you might recognize from Physics 212. The other limit is where L approaches zero and the line charge approaches a point charge. This limit is more subtle. If we set L=0 we get zero field. This makes sense since if L=0 the total charge L times lambda vanishes and you get no E-field. A better approach is to start by multiplying Lambda L to get the total charge q and then set L=0 in the sqrt(L^2+4h^2) = 2h. Alternatively you can expand L/sqrt(L^2+4h^2) to lowest non-vanishing order in L. You can either do this with a formal Taylor expansion or write sqrt(L^2+4h^2) using the binomial expansion theorem: sqrt(L^2+4h^2)=(2h)*[1+(L/2h)^2]^(1/2) approximately = (2h)*[1+L^2/8h^2] which is second order in L. Thus in the vanishing L limit we have Coulomb’s constant times q/h^2 which is the correct field for a point charge.

Gauss's Law

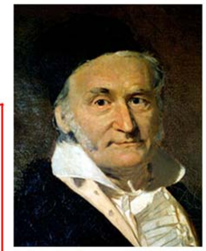
Brute force superposition often involves an ugly integral. In Physics 212 you learned a much easier way for symmetric cases.

$$\varepsilon_0 \int \vec{E} \bullet d\vec{a} = Q_{enc}$$



$$\varepsilon_0 \int \vec{E} \bullet d\vec{a} = \varepsilon_0 4\pi r^2 E_r = Q$$

$$\rightarrow E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (r > R)$$



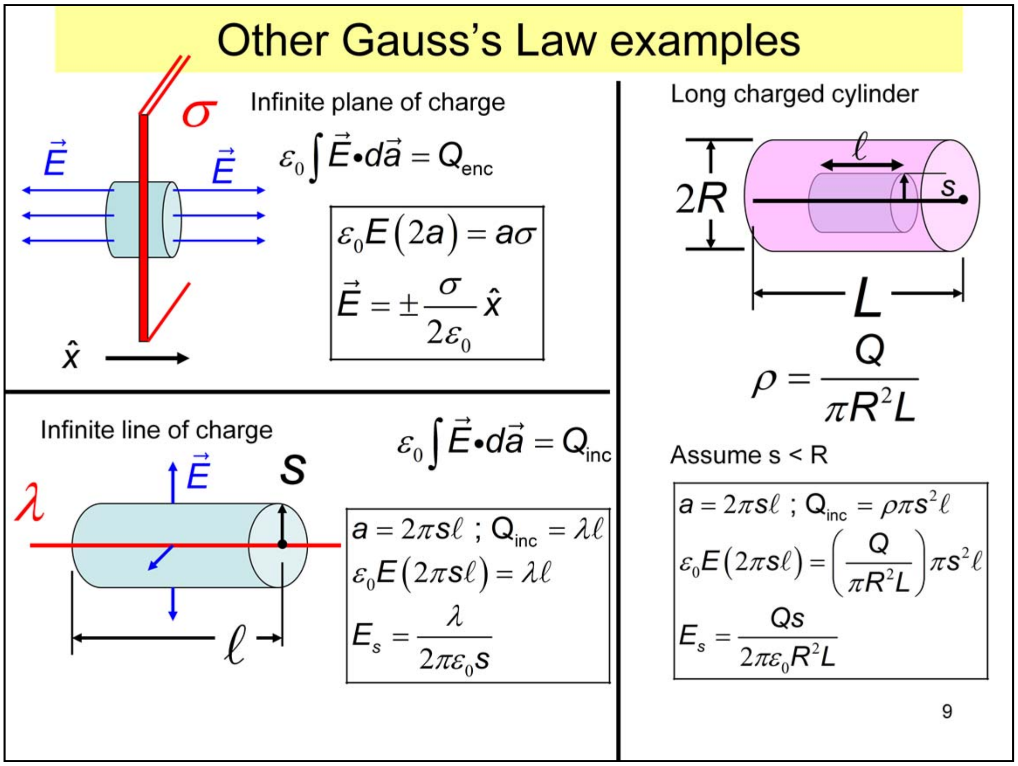
$$Q_{enc} = \rho\tau = \rho\left[\frac{4\pi r^3}{3}\right] ; \rho = \frac{Q}{4\pi R^3 / 3}$$

$$\varepsilon_0 4\pi r^2 E_r = \rho\tau = \left(\frac{Q}{4\pi R^3 / 3}\right)\left[\frac{4\pi r^3}{3}\right] = Q\left(\frac{r^3}{R^3}\right)$$

$$\rightarrow E_r = \frac{Qr}{4\pi\varepsilon_0 R^3} \quad (r < R)$$

Two Gaussian spheres for $r > R$ and $r < R$

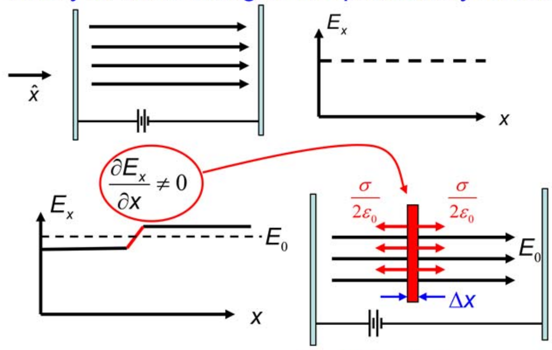
The forgoing superposition approach to computing the E-field even for one of the simplest problems is a bit tedious and usually requires consulting integral tables. Often direct superposition is the only way to compute the E-field as it was in our finite charge line case. But if a problem has sufficient symmetry we can find the field using Gauss's Law. Although we will show that Gauss's Law can be "derived" from Coulomb's law, we will think of it as a **fact** for a while. Gauss's law references a "bounding" surface around a volume of space and says that the surface integral of the E- field times epsilon_0 is the total charge enclosed by the "bounding" surface. Since the right-hand side of Gauss's law is the scalar Q (ie no direction) and E is a vector one needs to dot it into the surface vector on the left side of Gauss's to get a sensible relation. The surface vector element is the "outward" normal to the surface and has a "length" equal to the area of the element. To remind you how to use Gauss's Law we consider the E-field from a uniform (constant density) sphere of charge of radius R and consider the E-field both inside and outside the sphere. The first step in applying Gauss's law is to find a "suitable" Gaussian surface. This is usually a surface whose normal points in the direction of the E-field and a surface where the magnitude of the E-field is constant. If no such surface can be found, Gauss's Law – while still true– is of little use and one must use superposition or some other trick to find the E-field. In this case sphere's centered on the charge center will work since the E-field points radially outwards and is normal to the sphere surface and the field only depends on radius and thus E is constant over the surface. This means the surface integral is just the area of the surface times E (which is written as E_r since E only has a radial component. Gauss (1777-1885) was the mathematician who developed the divergence theorem which is the basis of Gauss's law along with a myriad of other contributions to physics and statistics.



Here are some hopefully familiar versions of the classic Gauss's Law prototypes. The upper left is the infinite plane of charge that holds a constant charge per unit area of sigma. We use the "pill-box" (squat cylinder) Gaussian surface. The only surface integral (or electric flux) contributions are from the two circular ends of the pill-box since the are transverse to the E-field (ie both surface normals are parallel to the E-field). The cylindrical surface is parallel to the E-field, has normals that are transverse to the fields and thus E dot a vanishes and thus do not contribute to the flux. Beneath the plane case is the infinite wire which we essentially calculated by brute force. Here we use a Gaussian cylinder of length L but this time the cylindrical surface contributes and the ends do not. Following the Griffiths notation we use "s" as the radial cylindrical coordinate (the rest of the world calls this "rho" but rho is a confusing choice for Gauss's Law since its also the charge density). Finally we consider a long charged cylinder of length L and radius R. Outside the cylinder we will get the same answer as the line of charge with lambda = Q/L. Inside the cylinder we use a cylinder of radius s < R as our Gaussian surface. Interestingly enough we get a field that is proportional to s. You can easily verify that the outside and inside solutions agree at s=R.

Searching for charge in 1-d

Can you tell if charges are present by looking at E- fields?



$$\frac{\partial E_x}{\partial x}\Delta x = 2\times\frac{\sigma}{2\varepsilon_0} = \frac{\Delta q/(\Delta y \Delta z)}{\varepsilon_0} \rightarrow \varepsilon_0 \frac{\partial E_x}{\partial x} = \frac{\Delta q}{\Delta x \Delta y \Delta z} = \rho$$

Our 1-d : $\varepsilon_0 \frac{\partial E_x}{\partial x} = \rho(x)$ logically (eg isotropically) generalizes to:

$\varepsilon_0\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)=\rho(\vec{r})$ written as $\boxed{\varepsilon_0 \vec{\nabla}\bullet\vec{E}=\rho}$ w/ $\vec{\nabla}=\left(\frac{\partial}{\partial x}\quad\frac{\partial}{\partial y}\quad\frac{\partial}{\partial z}\right)$

This is the differential form of Gauss's Law.

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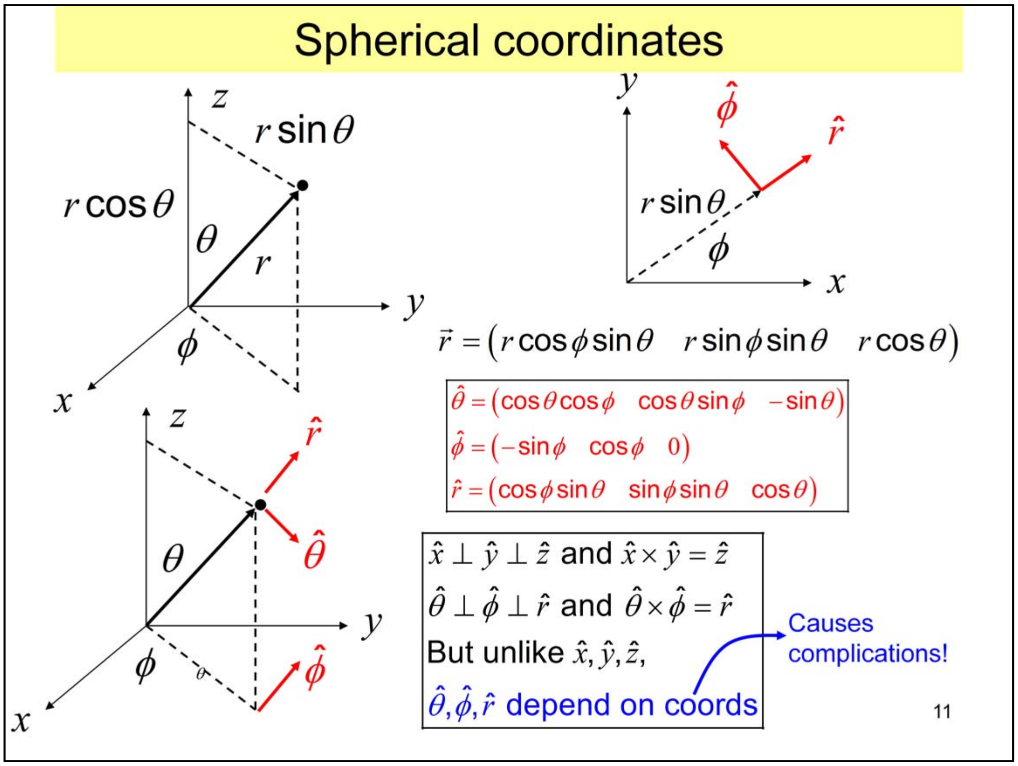
Imagine you are in a region between the plates of a parallel plate capacitor. As you know from Physics 212 or can easily work out from Gauss's Law, the field between "infinite" capacitor plates is uniform and in the direction normal to the plates which we will call the x-hat direction. So E_x will be constant and proportional to the voltage between the plates. What if you thought you were this region and found that E_x wasn't constant but had a small discontinuity indicated by the red region? One possible explanation was that you had encountered a thin slab of charge and were observing the superimposed of the field from this slab of charge and the much larger field from the capacitor. In the region before reaching the slab you see a smaller field than that of the capacitor and beyond the slab you see a larger field and (as you show in homework) the field has a constant slope while in the slab. The difference between E_x above the slab and below the slab will be twice the field of the equivalent charge plane which as we saw is sigma/(2 epsilon_0). This difference is (partial E_x/partial x) delta x. We can re-arrange this expression to get epsilon_0 times the slope of the E-field is the surface charge density divided by the thickness which is just the charge density rho. This is the one dimensional version of the differential form of Gauss's law. The "obvious" 3-dimensional generalization of this example is to include derivatives in each dimension. A neat notation for this 3 dimension derivative uses the inverted triangle "del" . We think of del as a vector with the three indicated coordinates, We will see how nifty this notation is and how frequently it generalizes. The divergence theorem discovered by Gauss links the integral version of Gauss's law (Physics 212) with this differential version.



We will frequently use curvilinear coordinates in Physics 435 and 436. These are coordinate systems with position dependent unit vectors. The big advantage of Cartesian coordinates is the unit vectors are in the same direction at any location. But for systems with cylindrical or spherical symmetry – curvilinear coordinates are usually the coordinates of choice. An example of a curvilinear system is spherical coordinates which I illustrate here. A particle is located by three coordinates, r , theta, and phi. The r coordinate is the distance from the origin, the theta coordinate is the angle the particle makes with respect to the z axis. The phi coordinate gives the angle between the particle and the x axis when one "projects" the particle to the x-y plane. I give you the Cartesian coordinates of a particle whose position is specified by r, theta, and phi. I hope you can understand this expression through a combination of visualization and trigonometry. I also show the three spherical unit vectors. You note they point in the direction of increasing r (r-hat), increasing theta (theta-hat), and increasing phi (phi-hat). At any given point, they are mutually perpendicular but their direction at a particle clearly depends on the particle's position. Using your right-hand you can easily see the various cross products such as theta-hat cross phi-hat = r-hat and circular permutations such as phi-hat cross r-hat = theta-hat. Using more visualization and trig you can confirm the Cartesian components of the spherical unit vectors. Incidentally they are written on the inside cover of Griffiths along with lots of other easy reference math.

The Spherical Divergence

Writing $\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho$ in Spherical coords.

$$\hat{r} = (\cos\phi \sin\theta \quad \sin\phi \sin\theta \quad \cos\theta)$$
$$\hat{\phi} = (-\sin\phi \quad \cos\phi \quad 0)$$
$$\hat{\theta} = (\cos\theta \cos\phi \quad \cos\theta \sin\phi \quad -\sin\theta)$$

$$\hat{x} = \cos\phi \sin\theta \, \hat{r} + \cos\theta \cos\phi \, \hat{\theta} - \sin\phi \, \hat{\phi}$$
$$\hat{y} = \sin\phi \sin\theta \, \hat{r} + \cos\theta \sin\phi \, \hat{\theta} + \cos\phi \, \hat{\phi}$$
$$\hat{z} = \cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}$$

1) Find Cartesian coords: $E_x = \vec{E} \cdot \hat{x} = \cos\phi \sin\theta E_r + \cos\theta \cos\phi E_\theta - \sin\phi E_\phi$

2) Transform derivatives with chain rule: $\dfrac{\partial}{\partial x} = \dfrac{\partial r}{\partial x}\dfrac{\partial}{\partial r} + \dfrac{\partial \theta}{\partial x}\dfrac{\partial}{\partial \theta} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial}{\partial \phi}$

3) Put it together and simplify.

$$\vec{\nabla} \bullet \vec{\mathrm{E}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta E_\theta\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\left(E_\phi\right)$$

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The divergence is particularly simple and memorable in Cartesian coordinates. The form in spherical coordinates (also on Griffiths' cover) is more complicated primarily since when you take a position derivative of a vector in curvilinear coordinates you need to take into account that unit vector is position dependent as well. How would you find spherical form? Here is a three part (albeit highly inelegant) method. First find E_x , E_y, and E_z from E_r, E_theta, and E_phi.

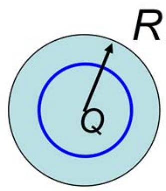
One way to do this is to write E-vector = r-hat E_r + theta-hat E_theta + phi-hat E_phi and then dot it into x-hat to find E_x. You can see that this was done to get my E_x expression. The derivatives can then be computed using the multidimensional version of the familiar chain rule. You can then add all three pieces of the Cartesian divergence together and you get the indicated form. More elegant methods are alluded to in Griffiths. Some of the extra pieces such as 1/r parts to the theta and phi divergence are clearly necessary on the basis of dimensional analysis.

Lets test out the spherical divergence

$$\varepsilon_0 \vec{\nabla}\bullet\vec{E} = \rho \qquad \vec{\nabla}\bullet\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\right) = \frac{\rho}{\varepsilon_0}$$

First the $r < R$ case

$$E_r = \frac{Qr}{4\pi\varepsilon_0 R^3} \quad (r < R)$$



$$\rho = \frac{Q}{4\pi R^3/3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{Qr}{4\pi\varepsilon_0 R^3}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\times 0\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(0\right) = \frac{\rho}{\varepsilon_0}$$

$$\frac{Q}{4\pi\varepsilon_0 R^3}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\right) = \frac{Q}{4\pi\varepsilon_0 R^3}\frac{1}{r^2}\left(3r^2\right) = \frac{3Q}{4\pi\varepsilon_0 R^3} = \frac{\rho}{\varepsilon_0} \rightarrow \rho = \frac{Q}{4\pi R^3/3}$$

It works!

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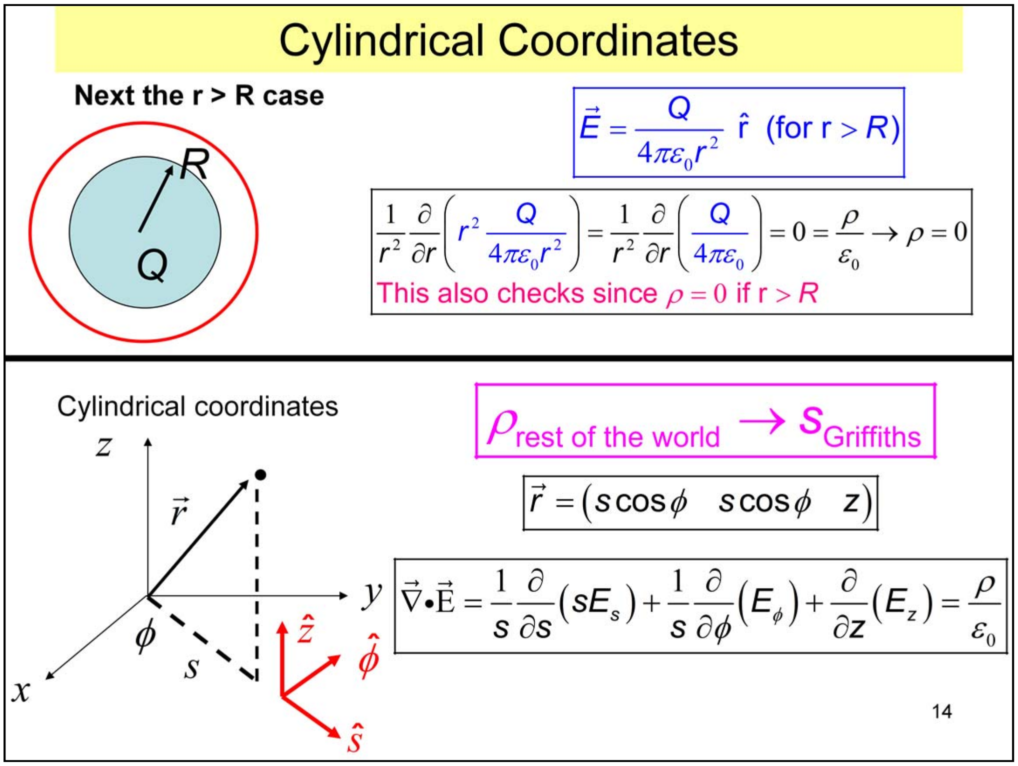
Here is a quick sanity check of the differential Gaussian law in spherical coordinates. Take the field inside of a uniform sphere of charge. The divergence of this field (del dot E) should be the charge density divided by epsilon_0. We plug our form into the spherical divergence formula. Since E only has an E_r component we only use the first term. We only need to differentiate r^3 with respect to r which is easy. We manage to get the right answer. A uniform ball of charge of radius R has 3Q/(4 pi R^3) as the volume density since (4 pi R^3/3) is the volume of a sphere.

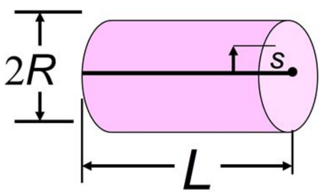


We next check the other case of the spherical ball, where we are outside of the ball with r > R. In this case we have the usual 1/r^2 radial E-field. If we stick this field into the radial part of the spherical divergence we get zero. It is interesting that a non-zero, non-constant field can have a zero divergence. But of course this makes sense since there is no charge once we are out of the ball of charge.

The other coordinate system that we will frequently use is the cylindrical coordinate system. Here we specify a position using s, phi , and z. s is the radial coordinate sqrt(x^2+y^2) , the same phi coordinate used in spherical coordinates, and the Cartesian coordinate z. The use of s for the cylindrical coordinate is a second Griffiths notation which is seldom seen outside of his text book. The cylindrical radial coordinate is often written as r (creating confusion w/ the spherical coordinate) or rho (creating confusion w/ the charge density). I show the Cartesian coordinates of a point specified by s, phi, and z which you should try to get by visualization and trig. I also show the three cylindrical unit vectors. When all the coordinate transformation tedium is performed you get the indicated form for the cylindrical divergence on Griffiths' cover.

Check of cylindrical divergence

$$\boxed{\vec{\nabla}\bullet\vec{\mathrm{E}} = \frac{1}{s}\frac{\partial}{\partial s}\left(sE_s\right) + \frac{1}{s}\frac{\partial}{\partial \phi}\left(E_\phi\right) + \frac{\partial}{\partial z}\left(E_z\right) = \frac{\rho}{\varepsilon_0}}$$



$$\rho = \frac{Q}{\pi R^2 L}$$

The $s > R$ form works:

$$E_s = \frac{Q/L}{2\pi\varepsilon_0 s}$$

$$\vec{\nabla}\bullet\vec{\mathrm{E}} = \frac{1}{s}\frac{\partial}{\partial s}\left\{s\left(\frac{\lambda}{2\pi\varepsilon_0 s}\right)\right\} = 0 = \rho$$

As does the $s < R$ case $\boxed{E_s = \frac{Qs}{2\pi\varepsilon_0 R^2 L}}$

$$\vec{\nabla}\bullet\vec{\mathrm{E}} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{Qs}{2\pi\varepsilon_0 R^2 L}\right) = \frac{1}{s}\frac{Q}{2\pi\varepsilon_0 R^2 L}\frac{\partial s^2}{\partial s} = \frac{\rho}{\varepsilon_0}$$

$$\to \rho = \frac{Q}{\pi R^2 L}$$

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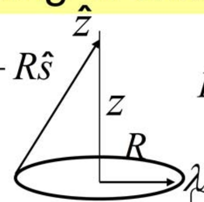
It is very easy to check the other case of being inside or outside of a long, charged cylinder. In both cases we get exactly the correct charge density rho.

Ring of charge using cylindrical coords



$$\vec{\mathscr{r}} = z\hat{z} - R\hat{s}$$

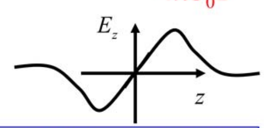
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\oint \lambda d\ell \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{1}{4\pi\varepsilon_0}\oint \frac{(z\hat{z} - R\hat{s})\lambda d\ell}{\sqrt{(z\hat{z} - R\hat{s})\cdot(z\hat{z} - R\hat{s})}^3}$$

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi} R d\phi \frac{(z\hat{z} - R\hat{s})}{\sqrt{z^2+R^2}^3} \neq \left\{ \begin{array}{l} \cancel{\frac{\lambda}{4\pi\varepsilon_0} 2\pi R \left[\frac{z\hat{z} - R\hat{s}}{\sqrt{z^2+R^2}^3}\right]} \quad \text{Since } \oint \hat{s}\, d\phi = \int_0^{2\pi} d\phi \begin{pmatrix} \hat{x}\cos\phi \\ +\hat{y}\sin\phi \end{pmatrix} = 0 \\ \frac{\lambda}{4\pi\varepsilon_0} 2\pi R \left[\frac{z\,\hat{z}}{\sqrt{z^2+R^2}^3}\right] \end{array} \right\}$$

Two checks

#1 limiting case $\vec{E} = \frac{\lambda}{4\pi\varepsilon_0} 2\pi R \left[\frac{z\hat{z}}{\sqrt{z^2+R^2}^3}\right] \xrightarrow{z \gg R} \hat{z}\frac{[2\pi R\lambda]_Q}{4\pi\varepsilon_0 z^2}$

#2 zero divergence

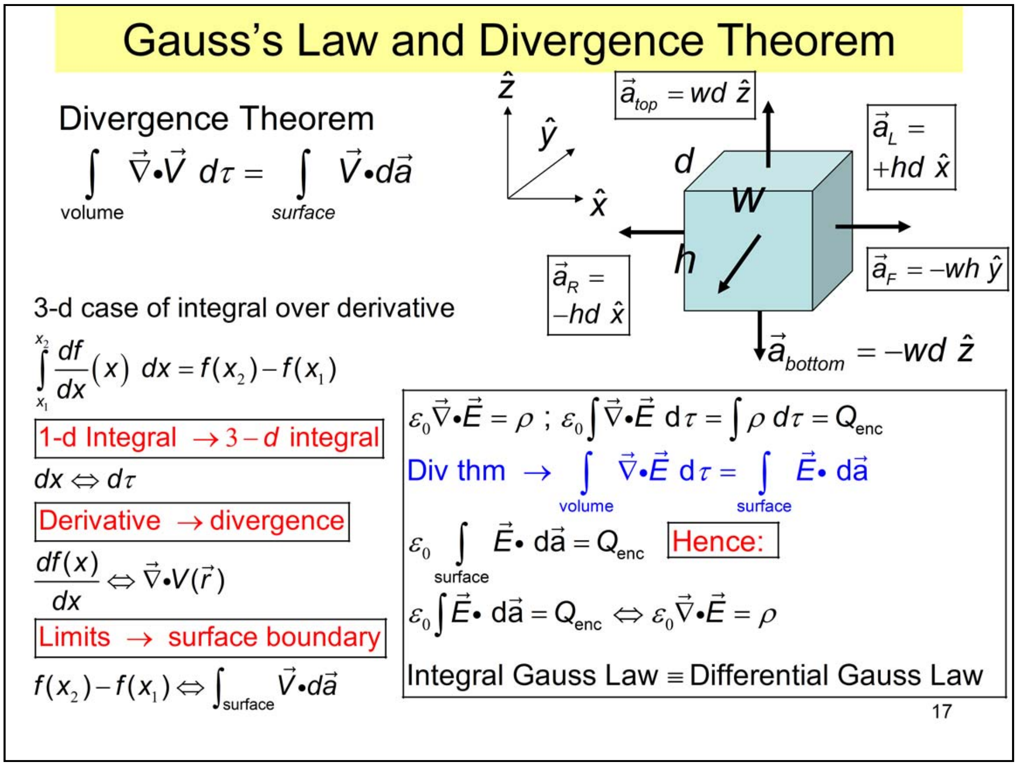


$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = 0 \rightarrow \frac{\partial E_z}{\partial z} = 0 \rightarrow \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{z^2+R^2}^3}\right) = 0??$$

Why did this check fail??

16

Often one uses curvilinear unit vectors when using Coulomb's law calculation to calculate E. This exercise illustrates some of the pitfalls in using curvilinear coordinates in a field integral. Here we consider the electrical field for a point on the symmetry axis of a uniform ring of charge of radius R with a linear charge density lambda. The relative displacement from the source to the observer or Griffiths r can be written as z z-hat –R s-hat for an the observer on the z-axis. We can write the E-field as an integral over phi. The z-hat unit vector is constant and so we can move it past the phi integral along with the R and z dependant terms. One might be tempted treat the s-hat unit vector as a constant and move it past the integral as well. But this gives the wrong answer since s-hat is a function of phi and the integral of s-hat over phi is zero. The moral is that one must convert the curvilinear unit vectors to Cartesian unit vectors prior to trying to integrate. We finally obtain a simple answer where the curvilinear coordinates did most of the work. We can check our result in the limit of z >>R where we should approach Coulomb's law for a point charge of Q = 2 pi Lambda R and we do. Another check is that for all of the points on the z-axis have zero charge density and hence Del dot E should equal zero. Since only the z-component of E exists this implies partial E_z/partial z = 0 which means E_z must be constant according to the differential form of Gauss's Law. But E_z varies as a function of z according to our integral and hence our solution seems to violate Gauss's law. Something is terribly wrong! The problem is that we only get E_x = E_y = 0 if x = y =0. To evaluate partial E_x/ partial x we need to be able to calculate vec E when we are off of the z-axis and our answer is only good for points on the z axis. Presumably the x and y derivatives kill the non-zero z derivatives and we will get a net zero divergence. But the x and y components will be require a more complicated calculation.



The mathematics that links the familiar Physics 212 form of Gauss's Law to the differential form is called the Divergence Thm obtained by Gauss. Griffiths makes the very interesting observation that the Divergence Theorem is really a three dimensional version of the fundamental thm of calculus which is the integral of the derivative is the function itself. Lets follow the interesting analogies. The one dimensional version says the integral of the derivative of a function involves the value of the functions at the two integration limits which we can think of as the "boundaries" of the integration domain. In the Divergence Thm the one dimensional integral becomes a 3-dimensional volume integral. The one dimensional derivative becomes the divergence. The function evaluated at the "boundary" of the domain becomes an integral of the function evaluated on the bounding surface The cube is a reminder on how to evaluate a surface integral with an emphasis on the nature of the surface integral. With the Divergence Theorem in hand, the bridge from the differential law to the integral law is a painless sprint. One does a volume integral on both sides of the divergence of E equals the charge density using an arbitrary volume. The integral of the divergence is the same as the surface integral over the "bounding" surface. The integral over the charge density is the charge "enclosed" by the bounding surface. Hence the surface integral over the field is essentially the charge enclosed by the surface. The integral and differential forms are equivalent. We are now well set up to show that Gauss's law follows from Coulomb's Law.

A paradox?

Consider $\vec{E}$ for point charge: $E_r = \dfrac{q}{4\pi\varepsilon_0 r^2}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{q}{4\pi\varepsilon_0 r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_0}\right) = 0 \rightarrow \frac{\rho}{\varepsilon_0}$$

Hence $\rho = 0$ everywhere?

But $\varepsilon_0 \int_{\text{sphere R}} \vec{E}\bullet d\vec{a} = \varepsilon_0 Q_{enc} \rightarrow$

$$\varepsilon_0 \frac{q}{4\pi\varepsilon_0 R^2}\times 4\pi R^2 = q$$

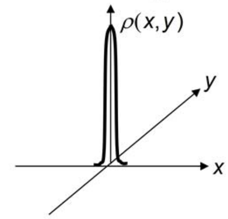
How can $\rho = 0$ everywhere but

$$\int_{\subseteq \text{all spheres}} \rho(\vec{r})\, d\tau = q \ ?$$

answer: $\dfrac{\rho}{\varepsilon_0} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(\dfrac{q}{4\pi\varepsilon_0}\right)$ is undefined

$\rho = 0$ is not valid at $r = 0$. We are ok if $\rho(r>0) = 0$ and $\rho(r=0) = \infty$

In 2-d this might look like this



The "fix" written as:

$$\vec{\nabla}\bullet\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3\left(\vec{r}\right)$$

$\delta^3\left(\vec{r}\right) = 0$ for $\vec{r} \neq \vec{0}$ and $\delta^3\left(\vec{0}\right) = \infty$

Normalized such that $\int_{\text{vol}\subseteq\vec{0}} \delta^3\left(\vec{r}\right) d\tau = 1$

The 4π then follows from Divergence Thm

$$\int_V \vec{\nabla}\bullet\frac{\hat{r}}{r^2}\, d\tau = \int_{r=R} \frac{\hat{r}}{r^2}\bullet\hat{r}\, da = \frac{\text{Area}}{R^2} = \frac{4\pi R^2}{R^2} = 4\pi$$

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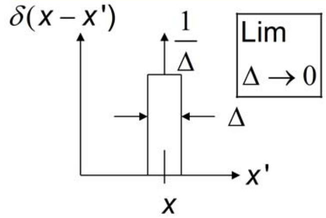
We start off with a mathematical paradox based on our two statements of Gauss's Law. Consider redoing the case of the E-field of a point charge. We form the spherical divergence of this field and find that it vanishes. No surprise here! We next "check" our result by the "equivalent" integral form of Gauss's law. The surface integral of the E-field centered around the point charge gives the q of the point charge. No surprise here as well. What's the problem? How can the charge density rho be zero everywhere and yet there is a point charge at the origin? I think the problem must be that our spherical divergence does not really "work" at the origin since it involves 1/r^2 terms that blow up at r=0. Perhaps rho is infinite at the origin? Indeed this makes sense since the integral of rho is always equal to q no matter how small in radius a sphere we pick. The only way this can happen is an infinite charge at our origin and no charge anywhere else. The 2-dimensional spike on the left side gives a useful way of visualizing the divergence of r-hat/r^2. At the origin our divergence goes to infinity . But it does it so that the volume of the spike is 1. In three dimensions we say the divergence of r-hat/r^2 is 4 pi times a three dimensional Dirac delta-function centered at the origin. As shown below the figure, the factor of 4 pi follows from applying the Divergence Theorem to the divergence of r-hat/r^2 which must be r-hat/r^2 evaluated on the surface of a sphere of radius R times the area of a sphere. We discuss the three dimensional delta-function next.

Delta-functions



We can write $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

The defining property of $\delta(x-x')$ is

$\int f(x')\delta^3(x-x')\ dx' = f(x)$

and our spike satisfies this since

$\int f(x')\delta^3(x-x')\ dx' = \lim_{\Delta\to 0} \int_{x-\Delta/2}^{x+\Delta/2} f(x')\frac{1}{\Delta}dx'$

$= f(x)\int_{x-\Delta/2}^{x+\Delta/2}\frac{1}{\Delta}dx' = f(x)\left[\frac{x'}{\Delta}\right]_{x-\Delta/2}^{x+\Delta/2} = f(x)$

The shifted version of $\vec{\nabla}\bullet\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$

becomes $\boxed{\vec{\nabla}_{(\vec{r})}\bullet\left(\frac{(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}\right) = 4\pi\delta^3(\vec{r}-\vec{r}\,')}$

and the defining property of $\delta^3(\vec{r}-\vec{r}\,')$

becomes $\int_{\vec{r}\subseteq V} f(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,')\ d\tau' = f(\vec{r})$

We can now obtain the differential form of Gauss's Law directly from Coulomb's Law since $\frac{(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}$ appears in Coulomb's Law

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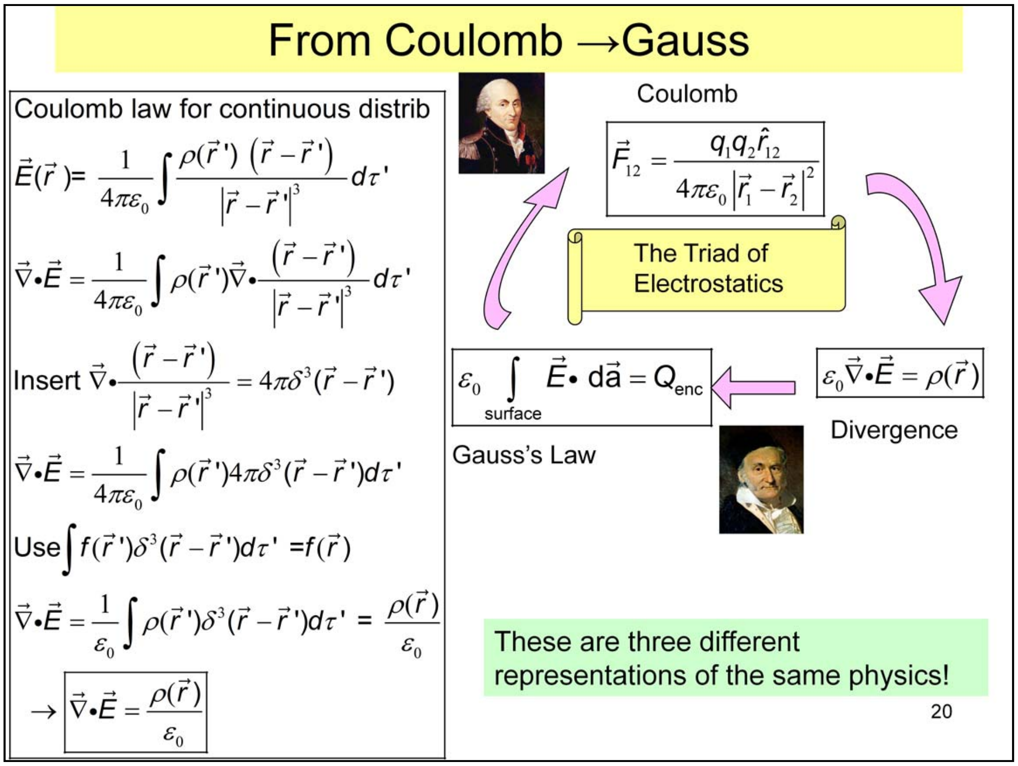
You may be familiar with the one dimensional delta function from mechanics. The function delta(x-x') can be visualized as a rectangular spike of width Delta and height 1/Delta centered about x in the limit that Delta approaches zero. With this definition, we can show that the integral of a function over delta(x-x') dx' is just f(x) as long as the limits of integration include the point x. The three dimensional delta function can be thought of as a product of three delta functions in x, y, and z.

We could shift the origin to a source point r' both on the left and right side of our divergence expression which is just equivalent to substituting r for r-r' The chief defining, property of the three dimensional delta-functions is that the integral of a delta function times a function gives the value of the function at where the argument of the delta-function is zero as long this zero point is in the integration volume. This makes intuitive sense since (in our three-dimensional example) the delta function only exists at the point r=r' and thus f(r) will need to be evaluated at r=r' to return something proportional to f(r'). The delta-function has been defined so that the proportional constant is 1. There are also 1 and 2 dimensions of the delta function which you may have seen in mechanics courses.



Armed with the 3-d delta function obtaining the differential form of Coulomb's law is again a very painless sprint. We write the E-field as an integral over the charge density. Recall we used this integral form to compute the E-field for a finite line of charge. We next compute the divergence of E by "dotting" del by the vector within the integral. In doing so its important to realize that the derivatives in the del are derivatives with respect to the coordinates of **observation** point (r) where the E-field is evaluated– not the derivative with respect to the source points r'. As such, del sails through the charge density rho(r') as if it were a constant and stops to operate on the (r-r')/|r-r'|^3 "Coulomb" piece. It returns 4 pi times our three-d delta function. The 4 pi part conveniently cancels the 4 pi in the denominator of Coulomb's constant. The integral over the charge density returns the charge density at the observation point. We are left with the differential form of Gauss's law. We have shown that the triad of electrostatics (1) Coulomb's Law (2) Gauss's integral form and (3) Gauss's divergence form are all equivalent when coupled with superposition. Hence we have the differential form of one of the four Maxwell equations.