Homework #11

1) Consider a resistor built out a cylinder of radius r which carries a current I . The material has a conductivity of σ_1 over the length ℓ_1 followed by a material with a conductivity σ_2 over the length ℓ_2 . Answer all parts of this problem in terms of $I, \sigma_1, \ell_1, \sigma_2, \ell_2$ and r . Assume the two electrodes are perfectly conducting plates of radius r.

   a) Find the resistance of the resistor

   b) Find the surface charge density between the first electrode and the start of the σ_1 material, at the junction of the σ_1 and σ_2 material, and between the σ_2 material and the second electrode. Find the total amount of surface charge Q_{tot} in these three regions. Hint-think Gauss's Law.

   c) Is there any volume charge density?

2) Work Griffiths problem 7.20 using $M_{ba} = \frac{\Phi_b}{I_a} = \frac{\oint \vec{A} \bullet d\vec{\ell}_b}{I_a}$ where $\vec{A}(\vec{r}) = \frac{\mu_0 \vec{m}_a \times \vec{r}}{4\pi r^3}$.

3) Consider two loops of radius "a" parallel to the x-y plane. Both centers are on the z axis but displaced by a distance Z. with $Z >> a$

   a) Calculate the mutual inductance between the two loops by assuming that one produces a magnetic dipole field at the center of the second loop and the second loop is so small that the flux is the area of the loop times the field. This part is a special case of Griffiths 7.20.

   b) Now calculate the mutual inductance using the Neumann formula. Here is how I did it. I parameterized a position vector on either loop by ϕ_1 and ϕ_2 I expanded $1/\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + Z^2}$ to the lowest non-vanishing order in $(x_1 - x_2)^2 / Z^2$ and $(y_1 - y_2)^2 / Z^2$ using a binomial expansion. The mutual inductance is then a double integral of the form $M \propto \int_0^{2\pi} d\phi_2 \int_0^{2\pi} d\phi_1 (\quad)$ which is slightly tedious but do-able.

4) A cylinder of radius R and length ℓ is aligned along the z axis. The cylinder contains a current of $\vec{J}(s < R) = J_0 \hat{z}$ and a surface current of $\vec{K}(s = R) = \frac{-J_0 R}{2} \hat{z}$ so that the net enclosed current is zero.

   a) Find $\vec{B}$ using Ampere's law.

b) Find the total stored energy using $U = \int \frac{\vec{B} \bullet \vec{B}}{2\mu_0} d\tau$.

c) Find $\vec{A}$ using $\oint \vec{A} \bullet d\vec{\ell} = \int \vec{B} \bullet d\vec{a}$ using a loop $\perp$ to $\vec{B}$. Assume $\vec{A} \propto \hat{z}$ and $\vec{A}(s=0) = \vec{0}$.

d) Show that the $\vec{A}$ you obtained in c) can be used to compute the stored energy using $U = \int \frac{\vec{A} \bullet \vec{J}}{2} d\tau \to \int \frac{\vec{A} \bullet \vec{J}}{2} d\tau + \int \frac{\vec{A} \bullet \vec{K}}{2} da$

5) Consider an infinitely long air-core solenoid of radius $s = R$. The symmetry axis of the solenoid is along the z-axis. A constant current would create a constant B field. In this problem you are asked to compute the electric and magnetic fields when a sinusoidal varying current flows through the solenoid. The fields can be written as a product of $\sin \omega t$ or $\cos \omega t$ and a Taylor series in increasing powers of $\frac{\omega s}{c}$. To zero order the magnetic field is given by $\vec{B}^{(0)} = \hat{z}\ B_0 \sin \omega t$. You should use the variables R, ω, B_0 and physical constants to answer all parts of this problem.

a) Use the zero order B-field expression to obtain a first order (in ω) expression for $\vec{E}(s,t)$. This is easy to do with Faraday's law.

b) Using the $\vec{E}(s,t)$ expression you obtained in a) to obtain $\vec{B}(s,t)$ to second order in ω.

c) Develop a double integral, consistency relation for $B_z(s)$. The approach is to write the $B_z(s)$ in terms of $\partial E_\phi / \partial t$ using the integral form of the Maxwell-Ampere law, and $E_\phi(s)$ in terms of $\partial B_z / \partial t$ using Faraday's law. Use the same loops you used in part a) and part b).

d) Use the double integral expression of part c) to obtain a recursion relation between the coefficients B_n in the expansion $\vec{B}(s,t) = \hat{z} \sin \omega t \sum_n B_n s^n$. The recursion relation shows how to obtain the coefficient B_{n+2} from the lower order coefficient (B_n). Write the $\vec{B}(s,t)$ expansion up to and including s^6 terms and sum it up if you recognize the series.