Laplace's Equation

Laplace's Equation is for potentials in a charge free region.

Pressure on conductors

Separation of variable solutions

Properties of Laplace solutions

Separation of variables in curvilinear coordinates

Method of images

Multipole expansions

monopole

$$V \propto \frac{P_0(\cos\theta)}{r}$$

dipole

$$V \propto \frac{P_1(\cos\theta)}{r^2}$$

quadrupole

$$V \propto \frac{P_2(\cos\theta)}{r^3}$$

octopole

$$V \propto \frac{P_3(\cos\theta)}{r^4}$$

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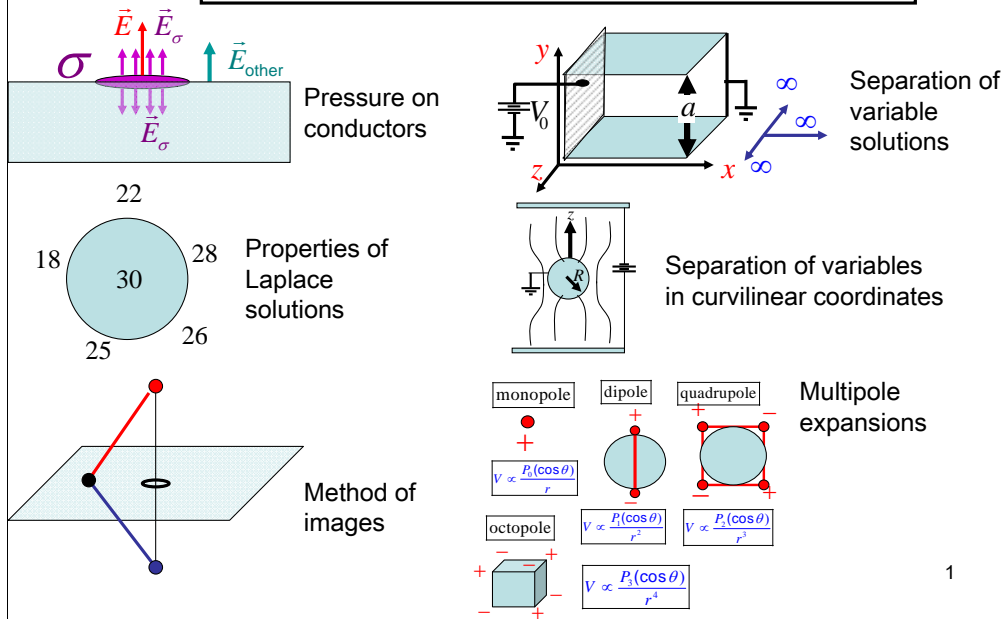
We will learn quite a bit of mathematics in this chapter connected with the solution of partial differential equations. In this case we will discuss solutions of Laplace's Equation which is used to find the potential as a function of position in charge free regions. Many of the same techniques we develop here, you will see again in the solution of the Schrodinger Eq. in quantum mechanics. We begin by describing the properties of perfect conductors including a discussion of the electrostatic pressure on a conductor. We turn next to a discussion of the formal properties of Laplace Equation including the illustrated averaging theorem , which leads to the uniqueness theorem. The Uniqueness theorem says that if one specifies the potential everywhere on the surface that bounds a region, one has a unique solution for the potential everywhere within the surface. This observation leads to the method of images which – for example– allows you to solve for the complicated potential present when a charge is placed over a grounded conducting plane. We turn next to the general "separation of variable" solution of Laplace's Eqn. This is the standard way of solving the crucial partial differential equations of physics. Separation of variables makes extensive use of the theory of orthogonal functions in order to match the necessary boundary conditions to give a unique, Laplace solution. You may have had some exposure to this in the form of Fourier Series. We then extend the separation of variables technique to spherical and cylindrical coordinate systems. We will develop solutions with certain features that very much resemble the electron cloud solutions of the Schrodinger equation. We conclude with a discussion of the multipole expansion which accurately describes the potential and fields a long distance from the charge. In particular the concept of an electric dipole plays a critical role in the interaction of electric fields with matter which is our next major topic.
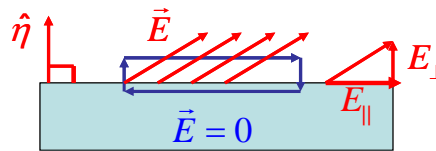
Conductors

We review some basic facts you know about
ideal conductors from Physics 212.

1. The electrical fields vanish within ideal conductors. The charges within the conductors re-arrange themselves to cancel all fields.

This means that the conductor is an equipotential since between any two points $\vec{r}_A$ and $\vec{r}_B$ in the conductor $V(\vec{r}_B) - V(\vec{r}_A) = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{E} \cdot d\vec{\ell} = 0$

Since there is no curl and E=0 inside conductor:

$$\nabla \times \vec{E} = 0 \rightarrow \oint \vec{E} \cdot d\vec{\ell} = 0 \rightarrow E_{\|} = 0$$

$$\rightarrow \vec{E} \parallel \pm \hat{\eta} \text{ or } \vec{E} \perp \text{ conductor}$$

$\hat{\eta}$ $\vec{E}$ $E_\perp$ $E_\|$ $\vec{E} = 0$

2. Since $\vec{E}$=0 within conductors, $\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = 0$. and hence no charges reside within a conductor and thus any charges can only exist on the surface of the conductor.

2

The goal of this chapter is to show you how to solve for the potential set up by various conductors. We begin by reviewing the properties of ideal (perfect) conductors which you may have learned in Physics 212. The first property is that the electrical field within an ideal conductor is zero. Essentially this is because a conductor offers no resistance to charge motion and hence the charges naturally re-arrange themselves to kill the electrical field and thus lower the net energy. If there is no electrical field in the conductor, the voltage difference between any two points on the conductor is zero which means the conductor is an equipotential. The vanishing of the electric field inside the conductor also implies that the parallel component of the electrical field just outside of the conductor vanishes meaning the E-field enters normally into the conductor. To show this we recall that the curl of an electrostatic field is zero, which means line integral of E around any complete path vanishes. We pick a path that straddles the conductor surface. Inside the conductor, the E-field vanishes and hence that leg of the path is zero. If the E-field came in at any angle other than normal, we would have a net line integral of E|| L which would violate the zero curl rule. There can be no volume charge within the conductor since rho is proportional to the divergence of the E-field and the E-field is zero. Of course if we charge the conductor with a spark the charge must go somewhere. Since it cannot appear inside, it appears on the surface and we can have a surface charge.
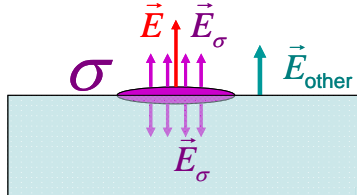
Surface charge on a conductor
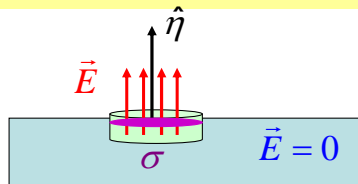
$\hat{\eta} \equiv$ outward normal

$$\varepsilon_0 \vec{E} \bullet \hat{\eta} a = Q_{enc} = \sigma a \rightarrow \boxed{\sigma = \varepsilon_0 \vec{E} \bullet \hat{\eta}}$$

Pressure on the conductor

$$\vec{F}_\sigma = (a\sigma)\vec{E}_{other}; \; \vec{E}_\sigma = \frac{\sigma\hat{\eta}}{2\varepsilon_0} = \frac{\varepsilon_0 \vec{E} \bullet \hat{\eta}\hat{\eta}}{2\varepsilon_0} = \frac{\vec{E}}{2}$$

$$\vec{E}_{other} = \vec{E} - \vec{E}_\sigma = \vec{E} - \frac{\vec{E}}{2} = \frac{\vec{E}}{2}$$

$$\left|\vec{F}_\sigma\right| = a\sigma\frac{\left|\vec{E}\right|}{2} = a\left(\varepsilon_0 \vec{E} \bullet \hat{\eta}\right)\frac{\left|\vec{E}\right|}{2} = a\frac{\varepsilon_0 \vec{E} \bullet \vec{E}}{2}$$

$$\rightarrow \boxed{\mathcal{P} = \frac{\left|\vec{F}_\sigma\right|}{a} = \frac{\varepsilon_0 \vec{E} \bullet \vec{E}}{2}}$$
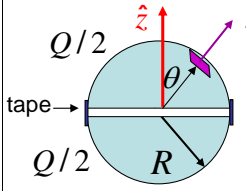
The force/area on the conductor is the field times the surface charge but we must use the field due to all other charges not the total field. We show this is ½ of the total field at the surface.

Interestingly the pressure equals the energy density at the surface. The force is is always directed outwards.   3

Actually if there is a normal electric field just outside a conductor, there **must** be a surface charge. If we know the field just outside the conductor at a spot we can find the surface charge at that spot using Gauss's law. We consider a small pill box with one end in the conductor and the other just outside the conductor. Gauss's law tells us the enclosed charge is just epsilon_0 times the electric flux passing through all sides of the pill box. The bottom of the pill box has no field and thus doesn't contribute to the flux. The sides don't contribute since their normal is parallel to the surface and the E-field is only perpendicular. Only the top of the pill box in the field free has a flux given by area eta dot E where eta points in the direction of the area vector or out of the conductor. The enclosed charge in the pill box is sigma area and hence we have a nice expression for sigma. We have found that there **must** be a surface charge if there is an electrical field just outside the conductor. In retrospect, this makes sense since the field due to the surface charge patch evidently cancels the external field inside the conductor and this is why conductors contain no E-field. Since we know the surface charge it is easy to find the pressure that the external field exerts on the surface. To get the force on this patch of surface charge we can can multiply the charge times the E-field. But there is a tricky complication. When one writes F = q E, one only considers the E-field due to all **other** charges but the charge whose force we are considering. A charge cannot exert a force on itself. Hence the F = (area) sigma E_other where E_other removes the self-field which we are writing as E_sigma. Fortunately E_sigma is easy to calculate from the same Gauss's law argument. If we consider a patch of surface charge in isolation we can find the fields through either face of a Gaussian pill box by setting epsilon_0 times the electric flux to the enclosed charge of (area) times sigma. We get the hopefully familiar result that E_sigma = sigma/(2 epsilon_0) where the factor of two comes about since now the top and bottom of pill box contribute to the flux. Since sigma = E/epsilon_0 we have the surprisingly general result that E_self = E/2 which means E_other is ½ of the external field. Using our result that sigma = E/epsilon_0 and E_other =E/2 we have a force of F= epsilon_0 E^2/2 (area) which means the pressure is given by epsilon_0 E^2/2 for all perfect conductors. The pressure expression is the same as the energy density of the field. I don't think this is a coincidence. If you recall from Physics 213, the energy of a fluid gas is given by the volume integral of the pressure or Int P d tau. Hence the pressure is the volumetric energy density just as it is here.

Pressure and the taped shell of charge

$Q/2$ $\hat{z}$ $\hat{r}$

Uniform (+) charge spread over metal shell held together w/ tape

tape →

$Q/2$ R

$$\Delta F_z = \frac{Q^2}{16\pi\varepsilon_0 R^2}\int_{\theta_1}^{\theta_2}\cos\theta\sin\theta\, d\theta$$

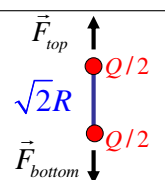Standard substitution of $u = \cos\theta$; $du = -\sin\theta\, d\theta$

$$\int_{\theta_1}^{\theta_2}\cos\theta\sin\theta\, d\theta = -\int_{\cos\theta_1}^{\cos\theta_2} u\,du = -\left(\frac{u^2}{2}\right)_{\cos\theta_1}^{\cos\theta_2} = \frac{\cos^2\theta_1 - \cos^2\theta_2}{2}$$

$$\boxed{\Delta F_z = \frac{Q^2}{32\pi\varepsilon_0 R^2}\left(\cos^2\theta_1 - \cos^2\theta_2\right)}$$

The field on surface is $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{r}$

thus $\mathcal{P} = \varepsilon_0\dfrac{\vec{E}\bullet\vec{E}}{2} = \dfrac{Q^2}{32\pi^2\varepsilon_0 R^4}$

A force element for charge at θ is

$$d\vec{F} = \frac{Q^2}{32\pi^2\varepsilon_0 R^4}\,da = \frac{Q^2}{32\pi^2\varepsilon_0 R^4}\left(R^2\sin\theta\, d\theta\, d\phi\right)\hat{r}$$

If integrate over all ϕ only F_z will survive

$$dF_z = \hat{z}\bullet d\vec{F} = \left|d\vec{F}\right|\hat{z}\bullet\hat{r} = \frac{Q^2\sin\theta\, d\theta\, d\phi}{32\pi^2\varepsilon_0 R^2}\cos\theta$$

The total F_z from $\theta_1 < \theta < \theta_2$ is

$$\Delta F_z = \frac{Q^2}{32\pi^2\varepsilon_0 R^2}\int_{\theta_1}^{\theta_2}\cos\theta\sin\theta\, d\theta\int_0^{2\pi}d\phi$$

$$\rightarrow \Delta F_z^{top} = \frac{Q^2}{32\pi\varepsilon_0 R^2}\left(\cos^2 0 - \cos^2 \pi/2\right) = \frac{Q^2}{32\pi\varepsilon_0 R^2}$$

$$\Delta F_z^{bottom} = \frac{Q^2}{32\pi\varepsilon_0 R^2}\left(\cos^2 \pi/2 - \cos^2 \pi\right) = -\frac{Q^2}{32\pi\varepsilon_0 R^2}$$

$$F_z = \pm\frac{Q^2}{32\pi\varepsilon_0 R^2} = \pm\frac{(Q/2)^2}{4\pi\varepsilon_0\left(\sqrt{2}R\right)^2}$$

Tape holds together this equivalent system

$\vec{F}_{top}$ $Q/2$

$\sqrt{2}R$

$\vec{F}_{bottom}$ $Q/2$ 4

Here is an application of the pressure law which I hope you will find interesting. If we charge a thin spherical shell, there will be a tendency for it to try to blow apart due to the Coulomb repulsion from the charges. We will use the pressure formula to compute the repulsive forces that acts on the tape that holds the two halves together. We define a z axis on the "poles" where the tape is on the sphere's "equator". The pressure involves the electrical field at the surface of the sphere. This is just a Coulomb's law field as you can easily verify with a Gauss's Law argument We will assume Q>0 but since the pressure is proportional to the square of the E-field, we would get the same bursting force for a negative charge. The pressure on the sphere is easy to calculate. An element of the force will be the pressure times an infinitesimal area da. The force will point outwards from the surface which is in the r-hat direction. A spherical area element is R^2 sin(theta) d(theta) d(phi). Since the x and y unit vectors depend on cos(phi) and/or sin(phi) they will both disappear when we integrate over phi. This leaves only the F_z component of the force which involves computing z-hat dot r-hat. These are two unit vectors which make an angle of theta with respect to each other so z-hat dot r-hat = cos(theta). We next write an integral for the F_z force acting on a piece of the sphere from theta_1 to theta_2 and all phi. We are left with an integral over theta. The first trick to try when doing a spherical integral over theta is the variable substitution u = cos(theta). This is because du is proportional to sin(theta) d(theta) which is the spherical solid angle element. Using this trick the integral is very easy to do and we have a very simple expression for Delta F_z. We next apply this to the "top" shell (0 < theta_1 < pi/2) and the "bottom" shell (pi/2<theta_1< pi). We get equal and opposite F_z expressions. We can manipulate the force expression and see that it is equivalent to the repulsive Coulomb force between two charges of Q/2 separated by sqrt(2) R. In Physics 436 we will introduce the Maxwell Stress tensor which provides a simpler and more elegant way of computing forces on a charge distribution such as this problem.

Potentials in a field free region

When no charge is present and $\vec{\nabla} \times \vec{E} = 0$

$\vec{E} = -\vec{\nabla} V$ and $\vec{\nabla} \bullet \vec{E} = \dfrac{\rho}{\varepsilon_0} = 0$

$\vec{\nabla} \bullet \vec{\nabla} \ V = 0$; $\left(\dfrac{\partial}{\partial x} \quad \dfrac{\partial}{\partial y} \quad \dfrac{\partial}{\partial z} \right) \bullet \left(\dfrac{\partial}{\partial x} \quad \dfrac{\partial}{\partial y} \quad \dfrac{\partial}{\partial z} \right) V = 0$ or $\nabla^2 V = 0$

$$\left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} \right) = 0 \text{ where V(x,y,z)}$$

This is called Laplace's Equation

Before showing how to solve this equation we develop an important property which leads to the method of images.
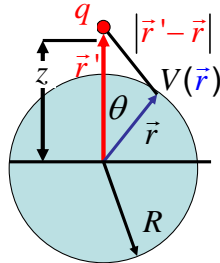
A unique solution to Laplace's equation exists once the potential is specified on a surface surrounding a charge free region.

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Laplace's equation is the equation for an electrostatic potential in a charge free region. We simply write E as a gradient of the potential and then (since there is no charge) we set the divergence of E to zero. This gives us a form where the del operator is dotted into the del operator. We are left with a very simple equation where the sum of the partial double derivatives acting on the potential is zero. The solutions to Laplace's Eqn. have some very nice properties which we can exploit to find solutions.

Laplace Averaging Theorem

We show average of potential over a sphere equals potential in center for an external charge

$$\left|\vec{r}-\vec{r}\,'\right|^2 = \left(\vec{r}\,'-\vec{r}\right)\cdot\left(\vec{r}\,'-\vec{r}\right)$$

$$\left|\vec{r}-\vec{r}\,'\right|^2 = \vec{r}\,'\cdot\vec{r}\,'-2\vec{r}\,'\cdot\vec{r}+\vec{r}\cdot\vec{r}$$

$$\left|\vec{r}-\vec{r}\,'\right|^2 = z^2 - 2rz\cos\theta + R^2$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}\,'\right|}$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0\sqrt{z^2 - 2rz\cos\theta + R^2}}$$

$$\left\langle V(\vec{r})\right\rangle = \frac{1}{4\pi R^2}\left(\frac{q}{4\pi\varepsilon_0}\right)R^2\int_0^{2\pi} d\phi\int_0^{\pi}\frac{\sin\theta d\theta}{\sqrt{z^2-2rz\cos\theta+R^2}}$$

$$\left\langle V(\vec{r})\right\rangle = \left(\frac{q}{\left(4\pi\right)^2\varepsilon_0}\right)2\pi\int_0^{\pi}\frac{\sin\theta d\theta}{\sqrt{z^2-2Rz\cos\theta+R^2}}$$

Let $\nu = z^2 - 2Rz\cos\theta + R^2$; $\dfrac{d\nu}{d\theta} = 2Rz\sin\theta d\theta$

$$\theta_{upper} = \pi \rightarrow \nu_{upper} = z^2 + 2Rz + R^2 = \left(R+z\right)^2$$

$$\theta_{lower} = 0 \rightarrow \nu_{lower} = z^2 - 2Rz + R^2 = \left(z-R\right)^2$$

$$\left\langle V(\vec{r})\right\rangle = \left(\frac{q}{2\left(4\pi\right)\varepsilon_0}\right)\frac{1}{2Rz}\int_{(Z-R)^2}^{(Z+R)^2}\frac{d\nu}{\sqrt{\nu}}$$

$$\left\langle V(\vec{r})\right\rangle = \left(\frac{q}{2\varepsilon_0}\right)\frac{1}{2Rz}2\left[\sqrt{\nu}\right]_{(z-R)^2}^{(z+R)^2}$$

$$\left\langle V(\vec{r})\right\rangle = \frac{q}{2\left(4\pi\right)Rz\varepsilon_0}\left[z+R-\left(z-R\right)\right]$$

$$\boxed{\left\langle V(\vec{r})\right\rangle = \frac{q}{4\pi\varepsilon_0 z} = V(\vec{r}=0)}$$

6

I will call our first general result the Laplace Averaging Theorem. This says that in electrostatic systems the average value of a potential over a spherical surface equals the value of the potential in the center of the sphere. It is actually easy to prove for a single charge and we could then use superposition to argue that it will apply to any charge distribution. I think it suffices to prove this for an exterior charge since the Laplace Eq. only applies to charge free regions. We align our z-axis in the direction from the sphere center to our charge. We next use the law of cosines to compute the distance between our "source" charge (r') and the "observation" point (r) where we are computing V(r). This will be a Coulomb law potential of the form q/|r'-r|. By the average of the potential over the sphere we mean the surface integral of V divided by the area of a sphere. Our z-axis choice makes the phi integral very simple and we are left with a rather ominous theta integral. Actually this integral (like many spherical integrals) is easy to do using a variable substitution. Rather than u = cos(theta) as we did on the last slide we use the even more convenient choice of setting nu equal to the piece under the radical. We are left with a very simple "power law" integral in nu. We are left with the Coulomb law potential for a point that lies a distance z from our charge. This is what we were trying to prove.

Laplace Extremum theorem

Lets say there is a local maxima in the Laplacean region that isn't on the boundary of the region. Here the local maximum is 30V.
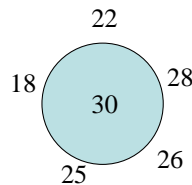
Hence both the largest and smallest value of the potential within a Laplacean region must lie on the boundary of the region.

Lets pick an averaging sphere where each point on a the surface is smaller than the 30 V maximum point.

But this **violates** the averaging theorem since the average of a set of numbers less than 30 V cannot equal 30.

The same argument excludes a local minimum

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It is now very easy to demonstrate another general theorem which I call the Laplace Extremum Thm. The Extremum Thm says that maximum and minimum of the potential in a region bounded by a surface must occur on the surface. Here is a demonstration. Imagine we have a 30 volt potential at some point. We draw a sphere surrounding the 30 V point and find that the potential of all points on this sphere have a voltage which is smaller than 30 volts. We know the average of all points on the sphere must equal the potential at the center. But the average of all our points is clearly smaller than 30 V and hence this situation violates the Averaging Thm. We thus conclude that the maximum and the minimum on our spherical surface must lie on the surface itself.

Uniqueness Theorem

If one specifies the potential on the boundary surface, the potential solution is unique.

$$V_3(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$$

Assume $V_1(\vec{r})$ and $V_2(\vec{r})$ are two different electrostatic solutions that both satisfy $V(\vec{r} \subseteq \text{surface}) = f(\vec{r} \subseteq \text{surface})$.

Construct $V_3(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$

$V_3(\vec{r})$ satisfies Laplace equation since:

$\nabla^2 V_3(\vec{r}) = \nabla^2 V_1(\vec{r}) - \nabla^2 V_2(\vec{r}) = 0 - 0 = 0$

$V_3(\vec{r} \subseteq \text{surface}) = 0$ since

$V_3(\vec{r} \subseteq \text{surface}) = f(\vec{r}) - f(\vec{r}) = 0$

Say there exists a point where

$V_1(\vec{r}_P) \neq V_2(\vec{r}_P)$ so $V_3(\vec{r}_P) \neq 0$

This $\vec{r}_P$ violates the extremum theorem for either $V_3(\vec{r}_P) > 0$ (left) or $V_3(\vec{r}_P) < 0$ (right)
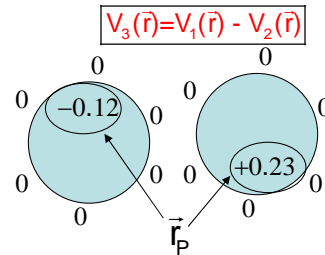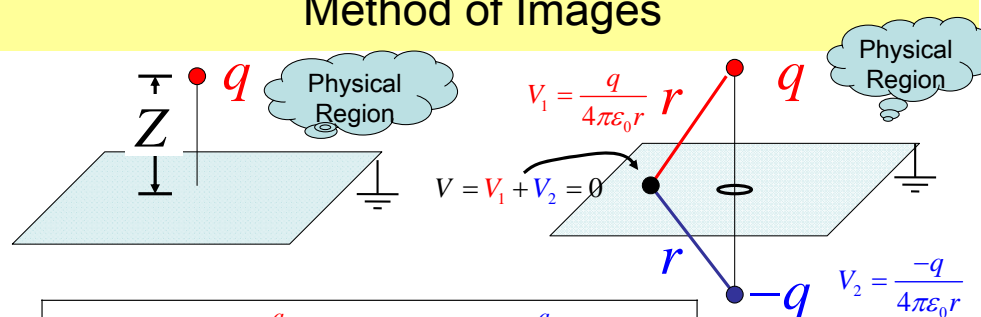
Hence the potential within a region is specified entirely by the potentials on the boundary surface

We can now establish one of the Laplace Uniqueness Theorems. This one says that if one specifies the potential on the boundary of a region, one gets a unique potential for every interior point. Again we use a Reductio ad Absurdium style demonstration. Lets say there are two different solutions for a potential V_1(r) and V_2(r) that have the same potential on a spherical boundary but a different values somewhere inside. We can construct a new potential V_diff(r) which equals the difference between these two solutions. This is also a solution to Laplace's Eq. since the Laplace Eq. is a linear DE and we know a linear combination of solutions is also a solution. Since the two solutions have the same "boundary" potential all points on the boundary have V_diff=0. By the Extremum Thm this means that the largest and smaller V_diff points within the region must be zero 0 as well. Any point where V_1 is not equal to V_2 violates the Extremum Thm. The basic idea is that if one finds a solution to Laplace's Eq. that works on the boundary – we know we have found the correct, unique solution. There are other "uniqueness" theorems that can specify enough boundary conditions to lead to a unique solution to Laplace's Eq. Some are discussed in your text.

q Physical Region

Z

$V_1 = \dfrac{q}{4\pi\varepsilon_0 r}$ r q Physical Region

$V = V_1 + V_2 = 0$

r

$-q$ $V_2 = \dfrac{-q}{4\pi\varepsilon_0 r}$

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-Z)^2}} + \frac{-q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z+Z)^2}} = 0$$

Hence we match the boundary and have the correct solution.

Lets compute the field and surface charge: $\vec{E} = -\vec{\nabla}V$

$$\frac{\partial}{\partial z}\left[x^2+y^2+(z-Z)^2\right]^{-1/2} = -\frac{1}{2}\left[x^2+y^2+(z-Z)^2\right]^{-3/2}\frac{\partial(z-Z)^2}{\partial z}$$

$$\frac{\partial}{\partial z}\left[x^2+y^2+(z-Z)^2\right]^{-1/2} = -(z-Z)\left[x^2+y^2+(z-Z)^2\right]^{-3/2}$$

simlar expressions for ∂/∂x and ∂/∂y

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\frac{(x\quad y\quad z-Z)}{\left[x^2+y^2+(z-Z)^2\right]^{3/2}} - \frac{(x\quad y\quad z+Z)}{\left[x^2+y^2+(z+Z)^2\right]^{3/2}}\right)$$

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The method of images is a great illustration of the power of the Uniqueness Thm. Say we are interested in computing the potential of a "physical" charge of carrying q placed a distance Z above a grounded metal plane. We will call the Z>0 region the "physical region" where we are interested in solving the Laplace Eq. Although it takes some imagining, we can think of the plane as forming the boundary around the physical region – perhaps think of the plane as the limiting case of a sphere of radius R around the point in the limit where R approaches infinity. By specifying that the plane is grounded, V = 0 for all points on the plane and we have thus specified the potential on all points of our "surface" and **any** solution which has V=0 on the plane will be the unique, correct solution. One way to match the V=0 BC is to place another "image" charge beneath the plane (in the non-physical region). If we use an image charge of –q and place it at –Z we can write an expression for V(x,y,z) with the property that V(x,y,0)=0 since the superimposed V expressions for the physical charge and image charge cancel on the plane and of course each satisfy Laplace Eq. since they are just Coulomb potentials. Astonishingly enough we are "finished" with the potential expression within the boundary (ie in the physical region). We have no clue as to the potential below the plane but perhaps we have no need to know. To have some fun with this solution, lets compute the induced surface charge on the plane. We know that sigma will be E_perp/epsilon_0 so we only need the E_z component but it is easy to get all three components which by taking taking the x,y, z derivatives of our potential expression.

Induced charge

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\frac{(x \quad y \quad z - Z)}{\left[x^2 + y^2 + (z - Z)^2\right]^{3/2}} - \frac{(x \quad y \quad z + Z)}{\left[x^2 + y^2 + (z + Z)^2\right]^{3/2}} \right)$$

$q > 0$

At $z = 0$, $E_x = E_y = 0$; $E_z = \frac{q\left[z - Z - (z + Z)\right]}{4\pi\varepsilon_0\left[x^2 + y^2 + Z^2\right]^{3/2}}$

$E_z = \frac{-2qZ}{4\pi\varepsilon_0\left[x^2 + y^2 + Z^2\right]^{3/2}} \rightarrow \vec{E}(x,y,0) = \frac{-2qZ\,\hat{z}}{4\pi\varepsilon_0\left[x^2 + y^2 + Z^2\right]^{3/2}}$

Lets compute the induced charge on the plate

$\sigma = \varepsilon_0\hat{\eta}\cdot\hat{E} = \varepsilon_0\hat{z}\cdot\hat{E} = \frac{-2qZ}{4\pi\left[x^2 + y^2 + Z^2\right]^{3/2}} = \frac{-2qZ}{4\pi\left[s^2 + Z^2\right]^{3/2}}$

$Q_{induced} = \int\limits_0^\infty \sigma 2\pi s\, ds = \frac{-qZ}{2}\int\limits_0^\infty \frac{ds^2}{\left[s^2 + Z^2\right]^{3/2}}$

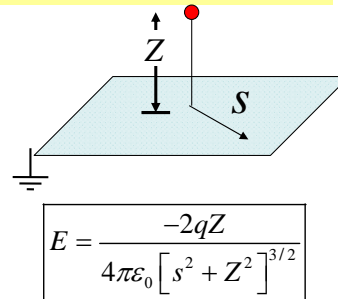$Q_{induced} = \frac{-qZ}{2}(-2)\left[\frac{1}{\sqrt{s^2 + Z^2}}\right]_0^\infty = -q$

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We now apply our E expression just above the plane at z = + infinitesimal. As you can see if you put z=0 the Ex and Ey expressions cancel and we are just left with a non-vanishing E_z. This had to happen since E must be normal to the surface just outside a conductor so it forms an important check. We can then get the surface density using sigma = E_z/epsilon_0. We see it has a simple form when cast in terms of the cylindrical variable s = sqrt(x^2 + y^2). We see sigma has the opposite sign as q and dies off as a function of s. It is easy to find the total induced charge by integrating sigma over the surface area of the plane which is just an integral over 2 pi s ds. This is very easy to evaluate if one simply switchs from s to s^2 and gives a very nice result that Q_induced = -q meaning that the total induced charge just cancels the physical charge.

Field pressure and force on plate

$$\mathcal{P} = \frac{\varepsilon_0 E^2}{2} = \frac{\varepsilon_0 \left(-2qZ\right)^2}{2\left(4\pi\varepsilon_0\right)^2 \left[s^2 + Z^2\right]^3} = \frac{q^2 Z^2}{8\pi^2\varepsilon_0 \left[s^2 + Z^2\right]^3}$$

$$F = \frac{q^2 Z^2}{8\pi^2\varepsilon_0} \int_0^\infty \frac{2\pi s\, ds}{\left[s^2 + Z^2\right]^3} = \frac{q^2 Z^2 \pi}{8\pi^2\varepsilon_0} \left(\frac{-1}{2}\right) \left[\frac{1}{\left[s^2 + Z^2\right]^2}\right]_0^\infty$$

$$F = \frac{q^2}{4\pi\varepsilon_0 \left(2Z\right)^2}$$

$$E = \frac{-2qZ}{4\pi\varepsilon_0 \left[s^2 + Z^2\right]^{3/2}}$$

$$F = \frac{q^2}{4\pi\varepsilon_0 \left(2Z\right)^2}$$

Same answer as for the image equivalent.

$2Z$
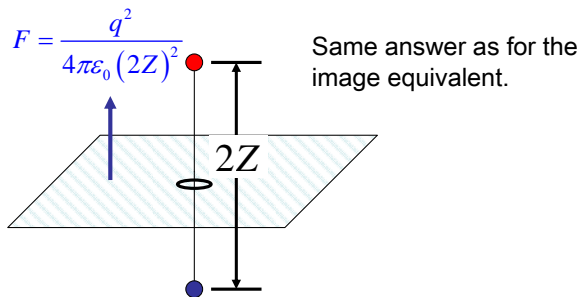
We also get a great chance to practice our "pressure" method for dealing with forces on a conductor. We first compute the pressure by squaring the E field just above the plane surface. We next integrate our pressure over the the area element 2 pi s ds

This problem is simpler than our taped sphere example since all of the forces point in the z direction. Again this is very easy to integrate using an s to s^2 variable substitution. We are left with the lovely result that the total force acting on the plane is exactly the same as the Coulomb attraction between the physical charge and the image charge (if it existed). By Newon's third law we know that the physical charge is attracted to the plane with an equal and opposite force.

The other classic "image" problem

q placed a from center of grounded sphere.

q' image charge is placed at b --out of physical region
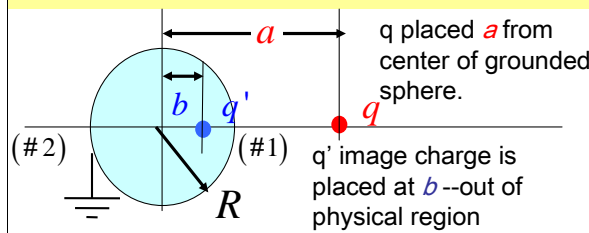
Lets use the image charge to compute the force of attraction between q and grounded sphere. **Perhaps** this will be correct.

Find b and q' that satisfy V=0 at #1 and #2

$$\boxed{\#1}\frac{q}{a-R}+\frac{q'}{R-b}=0 \;;\; \boxed{\#2}\frac{q}{a+R}+\frac{q'}{b+R}=0$$

$$q'=-q\frac{R-b}{a-R} \;;\; q'=-q\frac{b+R}{a+R} \rightarrow \frac{R-b}{a-R}=\frac{b+R}{a+R}$$

$$(R-b)(a+R)=(a-R)(b+R)\rightarrow \boxed{b=\frac{R^2}{a}}$$

$$q'=-q\frac{R-R^2/a}{a-R}\rightarrow \boxed{q'=-\frac{R}{a}q}$$

$$F=\frac{qq'}{4\pi\varepsilon_0\left(a-b\right)^2}=\frac{-q(qR/a)}{4\pi\varepsilon_0\left(a-R^2/a^2\right)^2}$$

$$F=-\frac{q^2}{4\pi\varepsilon_0}\frac{Ra}{\left(a^2-R^2\right)^2}$$

Using algebra one can show that the <u>entire sphere</u> is at V=0 not just points #1 and #2

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Here is another classic image problem for a physical charge placed a distance a from a grounded sphere of radius R. Here the boundary (eg sphere) "encloses" all of space outside the sphere. If a method of image solutions exists (not obvious) we need to find a location and charge magnitude that will cancel the potential at all points on the surface of either sphere or spherical shell. It is apparent we need a negative charge if q is positive and it needs to be out of the physical region (eg inside the sphere). A good guess might be to put the image charge along the radius vector that points to q. We don't know b or q' and we need two Eq. Any two points on the sphere will suffice. Points #1 and #2 seems like a good choice to simplify the algebra. We find that b is always within the sphere since b = R^2/a < R if a > R. This had to happen since the image charge is never in the physical region. And q' is negative and has a smaller magnitude than q. Of course this does not prove we have a valid solution. For that we need to show V=0 everywhere on the sphere – not just at the two points #1 and #2 which we used to get our b and q' values. It turns out V=0 everywhere but the algebra is not that illuminating. We can use our solution for V to compute the pressure on the sphere surface and integrate the radial component to find the total force on the sphere. It would be great if this were equal to the force of attraction between the physical and image charge. My understanding is that it is but there are no advanced guarantees that this will work.

In two dimensions Laplace's Eqn is:

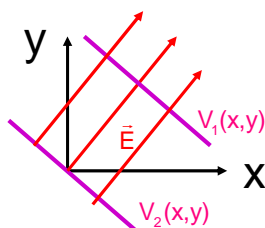$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x,y) = 0$$

$V(x,y) \propto xy$ also has $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial^2 V}{\partial y^2} = 0$

$$\frac{\partial^2 xy}{\partial x^2} = y\frac{\partial^2 x}{\partial x^2} = y\frac{\partial 1}{\partial x} = 0$$

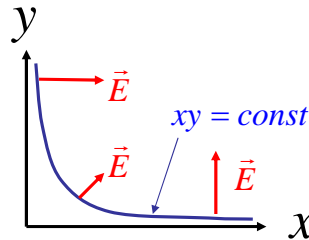One class of solutions have $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial^2 V}{\partial y^2} = 0$

One example is $V = \alpha x + \beta y + V_0$.

$\vec{E} = -\vec{\nabla}V = \begin{pmatrix} -\alpha & -\beta \end{pmatrix}$

y

$\vec{E}$

$xy = const$

$\vec{E}$ $\vec{E}$

x

$\vec{E}$

$V_1(x,y)$

$V_2(x,y)$

The hyperbolic potential produces a more interesting and useful field

This straight line potential is a valid but boring solution.

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We now discuss some more "mathematical" approaches to Laplace's Eq. To simplify, we will just consider two dimensions. One set of solutions are build out of functions which guarantee that that neither the x or y double derivative of V exists. One example is a linear potential. This is equivalent to a uniform E-field since the gradient of a linear V expression is constant. A somewhat more interesting one is V propto x y. Again the double derivatives of V are zero. Recall in the context of Laplace's Eq. the partial with respect to x means hold the other variables y and z constant and this is why the double derivatives vanish. If we set V=xy = constant we will form a hyperbola and hence the equipotentials will be hyperbolas. The fields are transverse to the equipotentials and I show a few examples. We previously saw an example of hyperbolic equipotentials in the previous chapter on potentials.

Exploring the hyperbolic potential — Fermilab

$+V_0$ $-V_0$

$2a$

$-V_0$ $+V_0$
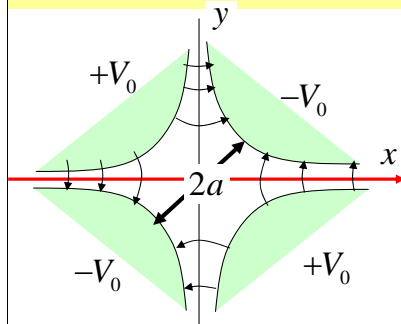
$xy = a^2$; $V = -V_0 \dfrac{xy}{a^2}$

Note $V(x,y)$ satisfies extremum condition.

$\vec{E} = -\vec{\nabla}V = \dfrac{V_0}{a^2}\vec{\nabla}xy = \dfrac{V_0}{a^2}\left(\dfrac{\partial xy}{\partial x}\quad \dfrac{\partial xy}{\partial y}\right) = \dfrac{V_0}{a^2}(y\quad x)$

$\left|\vec{E}\right| = \dfrac{V_0}{a^2}\sqrt{x^2+y^2} \propto \dfrac{V_0}{a^2}s$ $\qquad \sigma = \dfrac{V_0}{\varepsilon_0 a^2}s$

(E & σ increase further from center)

$\mathcal{P} = \dfrac{\varepsilon_0}{2}\left|\vec{E}\right|^2 = \dfrac{\varepsilon_0 V_0^2}{2a^4}s^2$

(pressure with increasing s)

Any charge in gap?

$\vec{\nabla}\bullet\vec{E} = \dfrac{V_0}{a^2}\left(\dfrac{\partial y}{\partial x} + \dfrac{\partial x}{\partial y}\right) = \dfrac{\rho}{\varepsilon_0} = 0.$ Indeed no charges are

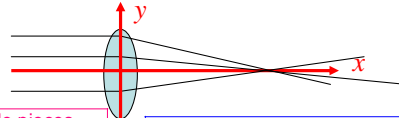present. Lets show $\vec{E} \perp \hat{\ell}$ where $\hat{\ell}$ is along electrode.
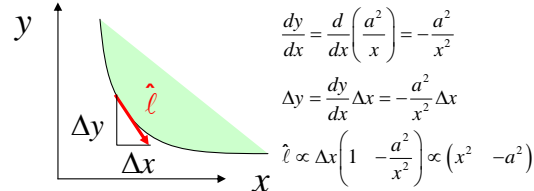
$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{a^2}{x}\right) = -\dfrac{a^2}{x^2}$

$\Delta y = \dfrac{dy}{dx}\Delta x = -\dfrac{a^2}{x^2}\Delta x$

$\hat{\ell} \propto \Delta x\left(1 \quad -\dfrac{a^2}{x^2}\right) \propto \left(x^2 \quad -a^2\right)$

$\hat{\ell}\bullet\vec{E} \propto \left(x^2 \quad -a^2\right)\bullet\dfrac{V_0}{a^2}(y\quad x) \propto x^2 y - a^2 x$

$\propto x(xy) - a^2 x \propto xa^2 - a^2 x = 0 \rightarrow \hat{\ell} \perp \vec{E}$

Hyperolic pole pieces can be used for magnets

$\vec{B} = \beta(y\quad x)$

$\vec{F}_y \propto \vec{B}_x \propto y$

Since y-bend $\propto$ y we have a magnetic lens used to focus beams at Fermilab

One example of the xy or hyperbolic potential is an electrostatic focusing element. We will write the voltage in the form V = -V0 xy/a^2 where "a" has length dimensions and V0 is in volts. The variable "a" is basically the aperture size. We include both V_0 and –V_0 hyperbolas and thus have 4 electrodes or four equipotentials. This is sometimes called a quadrupole lens in honor of the four electrodes. The lens part will be come clear shortly. We can take the gradient of V to find the E-field as a function of x and y. The charge density will be proportional to the square of the E-field just outside each electrode. We see that the field strength, surface charge density, and electrostatic pressure all grow as a function of s which is the distance from the central axis. The picture makes it clear why. As one gets further from the center, the gap between electrodes becomes smaller. The voltage is the average field times the separation and hence it must grow as the separation shrinks. We show this this graphically by drawing some field lines which should get denser as s increases. We next show that field lines are transverse to the equipotentials. We do this by drawing a dL vector along the hyperbola. Using derivative = rise/run we see this is proportional to the (x^2,a) vector and indeed L dot E is zero on the hyperbola. The reason this acts as a lens is because the field is proportional to the distance. This is the same thing that happens in a refractive lens. The lens bends the light proportional to the distance off axis. This means a set of parallel rays will be bent to converge on a single "focal" point. In our Magnetostatics Chp we will show how a quadrupole magnet can act to focus particle beams at an accelerator like Fermilab.
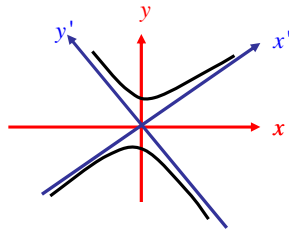
"Compensation" solutions

Have $\dfrac{\partial^2 V}{\partial x^2}, \dfrac{\partial^2 V}{\partial y^2} \neq 0$ but $\dfrac{\partial^2 V}{\partial x^2} = -\dfrac{\partial^2 V}{\partial y^2}$

A simple example is $V \propto \dfrac{x^2 - y^2}{2}$

$\dfrac{\partial^2 V}{\partial x^2} = 1$ & $\dfrac{\partial^2 V}{\partial x^2} = -1 \rightarrow \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

Rotation of same hyperbolic V

$V \propto \dfrac{x^2 - y^2}{2} = \left(\dfrac{x - y}{\sqrt{2}}\right)\left(\dfrac{x + y}{\sqrt{2}}\right) \propto x'y'$

Separation of variable solutions

This provides a systematic way of getting Laplace solutions from arbitrary boundary conditions.

Try solutions to $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

of the product form $V(x,y) = X(x)Y(x)$

$\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial^2}{\partial x^2}\{X(x)Y(y)\} = Y(x)\dfrac{\partial^2 X}{\partial x^2}$

since $\dfrac{\partial^2}{\partial x^2}$ only operates on x piece.

and $\dfrac{\partial^2 V}{\partial y^2} = \dfrac{\partial^2}{\partial y^2}\{X(x)Y(y)\} = X(x)\dfrac{\partial^2 Y}{\partial y^2}$

$\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = Y\dfrac{d^2 X}{dx^2} + X\dfrac{d^2 Y}{dy^2} = 0$

Note $\dfrac{\partial^2 X}{\partial x^2} \rightarrow \dfrac{d^2 X}{dx^2}$ since only 1 variable in X(x)

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Another class of solutions to Laplace's Eqn. are solutions where both the x and y double derivatives of the potential exist but exactly cancel. One example is V propto x^2 –y^2. We show that this satisfies Laplace's Eqn. but it is just a rotated version of the xy hyperbolic V that we were discussing. The other class of solutions – used for most Laplace problems – is a separation of variable solutions. The idea here is we write V as a product of one function of x only times another function of y only. Later we will write separation solutions which are functions of r and theta. What are the conditions on X(x) and Y(y) when we factorize the potential in the form V(x,y)=X(x)Y(y). The important point to realize is that Y(y) acts as a constant when we take a partial with respect to x and thus "sails through the derivative".

Separation of Variable Solutions

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

With some risk we can divide
both sides by X(x)Y(y) to get

$$\frac{1}{XY}\left(Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} \right) = \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

We note $\frac{1}{X}\frac{d^2 X}{dx^2} = f(x)$ and $\frac{1}{Y}\frac{d^2 Y}{dy^2} = g(y)$

But how can $f(x) + g(y) = 0$?

The only way is for f(x) and g(y) to be
cancelling constants: $f(x) = \lambda$; $g(y) = -\lambda$

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \lambda \text{ and } \frac{1}{Y}\frac{d^2 Y}{dx^2} = -\lambda$$

$$\frac{d^2 X}{dx^2} = \lambda X \text{ and } \frac{d^2 Y}{dy^2} = -\lambda Y$$

You might recognize these as either
exponential or sinusoidal cases depending
on sign of λ. For example if $\lambda > 0$
we can use $X(x) = \exp(\beta x)$

$$\frac{d^2 X}{dx^2} = \beta^2 \exp(\beta x) = \lambda \exp(\beta x)$$

Then $\beta^2 = \lambda$ which means

$$\frac{d^2 Y}{dy^2} = -\beta^2 Y \text{ and } Y = \sin(\beta y) \text{ is a solution.}$$

Of course $Y = \cos(\beta y)$ works as well.
Hence $Y(y) = A\sin(\beta y) + B\cos(\beta y)$ is the solution.
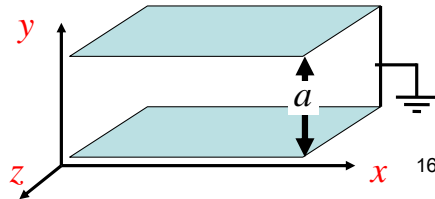Similarly X=exp(-βx) is an equally good solution

for $\frac{d^2 X}{dx^2} = +\beta^2 X \rightarrow X = C\exp(+\beta x) + D\exp(-\beta x)$

Hence in general:

V=$\left[D\exp(+\beta x) + C\exp(-\beta x) \right]\left[A\sin(\beta y) + B\cos(\beta y) \right]$

where A,B,C,D chosen to match boundary condition

Example:

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Hence we have a simple form for the Laplace's Eq. when we use factorizable potentials. The real magic comes about when we divide both sides of the Eq. by X(x)Y(y). This might be risky since we might be dividing by zero but no guts no glory. We are left with two terms that need to add to zero. One of these is the double derivative of X divided by X and the other is the double derivative of y divided by Y. You note we switched to ordinary derivative notation since both X and Y are only functions of one variable – namely x and y respectively so there is no need to talk about holding another variable constant. The key observation is that the double derivative of X/X can only be a function of x – say f(x). Similarly the Y double derivative over Y can only be a function of y – say g(y). But the only way f(x)+g(y) can add to 0 for all x and y is to have f(x) = constant and f(y) = -constant. Hence our partial differential equation "separates" into two ordinary differential equations in x and y. For our 2-d Laplacean case we use lambda as the "constant of separation" and our two differential equations are generalized exponentials (eg either exponentials or trig functions depending on the sign of lambda). We thus get solutions where X = A sin(lambda x) + B cos(lambda y) and Y = C exp(lambda y) + D exp(-lambda y) or vice versa. We would decide which coordinate is sinusoidal and which is exponential by the nature of boundary conditions and would choose the constants A,B,C, and D to satisfy these boundary conditions. Does this mean all solutions for the two dimensional Laplace Eq. consist of a sinusoidal time exponential factors? Yes and no. We can clearly superimpose the sinusoidal times exponential solutions for different separation constants (lambda) to match even complicated boundary conditions. Here is our first (pre-)example consisting of two grounded planes separated by a distance "a". They have infinite extent in z so our solutions will have V(x,y,z)=X(x)Y(y)Z(z) but Z(z) = constant and we can thus eliminate z as a variable and only consider X(x) Y(y).

Separation of Variable Example:

To go further we need more boundary conditions. Here is our first example.
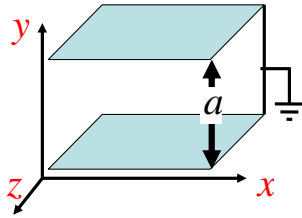
$$V=\left[D\exp(+\beta x)+C\exp(-\beta x)\right]\left[A\sin(\beta y)+B\cos(\beta y)\right]$$

We know $A\sin(\beta y)+B\cos(\beta y)=0$ for

$y=0$ and $y=a$

$A\sin(\beta 0)+B\cos(\beta 0)=D=0$
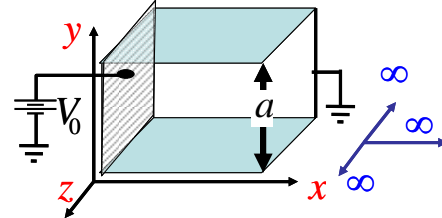
Hence $Y(y)=B\sin(\beta y)$

Now the upper plate BC

$$Y(a)=B\sin(\beta a)=0 \rightarrow \beta a=n\pi \rightarrow \beta=\frac{n\pi}{a}$$

where $n \subset \{0,1,2,3\cdots\}$

Hence

$$V=\left[D\exp\left(\frac{n\pi x}{a}\right)+C\exp\left(-\frac{n\pi x}{a}\right)\right]\sin\left(\frac{n\pi y}{a}\right)$$

$$V=\left[D\exp\left(\frac{n\pi x}{a}\right)+C\exp\left(-\frac{n\pi x}{a}\right)\right]\sin\left(\frac{n\pi y}{a}\right)$$

Since our BC extends to $x \rightarrow \infty$ we can't use

$D\exp\left(\frac{n\pi x}{a}\right)$ term. Thus

$$V=C\exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

But: $V(0,y)=C\sin\left(\frac{n\pi y}{a}\right) \neq V_0$???

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Here we use beta as the constant of separation and we must use the same constant for X(x) and Y(y) if we want to solve Laplace's Eq. The fact that we have "grounds" in the y direction gives us some useful guidance as to which coordinate to use for a sinusoidal solution and which coordinate to use for the exponential solution. If we use a sinusoidal solution we have the ability to put nodes at the potential zero or nodes at y=0 and y=a, this means our exponential solutions must be reserved for the X(x) functions. We want one node at y = 0 which we can get using X(x) = A sin(beta x) we can get a node at y =a by choosing beta = n pi/a where n is an integer such as 1,2,3,... We thus know our solutions are of the form V = [D_n exp(n pi x/a) + C_n exp(-n pi x/a)] sin(n pi y/a) and any integer n will work. We also have the freedom of a separate D_n and C_n constant for any n choice. To go further we need more boundary conditions – presumably in x. As a possible example, lets consider looking for solutions in the x > 0 region. To avoid our potential from blowing up at infinity , we evidently want a negative exponential which means our D_n constants must all be zero. Hence if we restrict our self to the region x>0 , 0 < y < a, and V(x,0)= V(x,a)=0 we know our solution is of the general form V(x,y) = sum_n=1,2,3,… C_n exp(-n pi x/a) sin(n pi y/a) since a superposition of any of our separation solutions will work. As you can see, as the boundary conditions are tightened we rapidly limit our separation solution forms. One possible BC would be to ground the x=0 plane Then V(0,y)=0 which means sum_n C_n sin(n pi y/a)=0 since exp(-n pi 0/a) = 1. Of course the only way to do this for arbitrary y is to set all C_n = 0 which means V(x,y) = 0 over the full region we are trying to represent. This is a fine solution which matches all of the BC as well as the Laplace Eq. But it isn't very interesting. It is just a three sided box with zero fields!
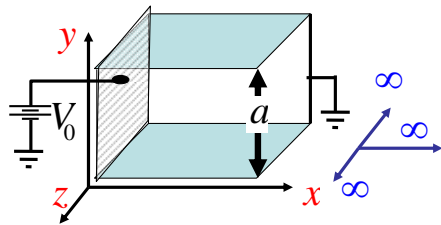
Fortunately we can match the BC since we have many possible n

$$V(0,y) = C\sin\left(\frac{n\pi y}{a}\right) \neq V_0$$

$$V_0 = \sum_{n=1,2,3\cdots} C_n \sin\left(\frac{n\pi y}{a}\right)$$

In general we can match any BC

$$V(0,y) = \sum_{n=1,2,3\cdots} C_n \sin\left(\frac{n\pi y}{a}\right) \text{ with}$$

a Fourier Series over $0 < y < a$.

$$\int_0^a V(0,y)\sin\left(\frac{n'\pi y}{a}\right)dy =$$

$$\sum_{n=1,2,3\cdots} C_n \int_0^a \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)dy$$

$$\int_0^a \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)dy = \frac{a}{2}\delta_{nn'}$$

Fourier

$$\int_0^a V(0,y)\sin\left(\frac{n'\pi y}{a}\right)dy = \sum_{n=1,2,3\cdots} C_n \frac{a}{2}\delta_{nn'} = \frac{a}{2}C_{n'}$$

or $C_n = \frac{2}{a}\int_0^a V(0,y)\sin\left(\frac{n\pi y}{a}\right)dy$

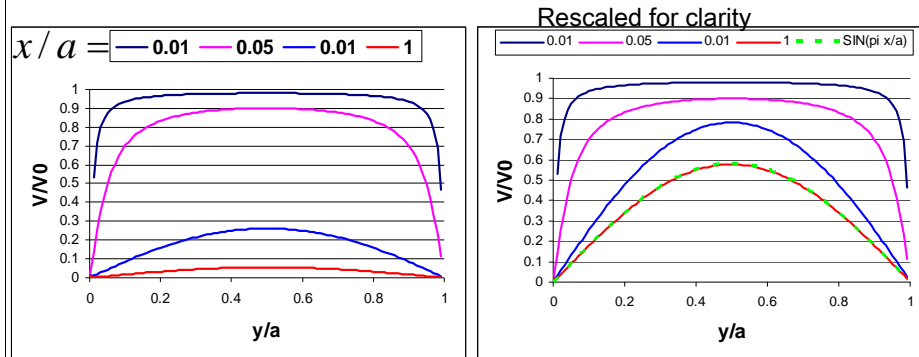and $V(0,y) = \sum_{n=1,2,3\cdots} C_n \sin\left(\frac{n\pi y}{a}\right)$

In our case $V(0,y) = V_0$

$$C_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = -\frac{2V_0}{n\pi}\left[\cos\left(\frac{n\pi y}{a}\right)\right]_0^a$$

$$C_n = \frac{4V_0}{n\pi} \text{ for } n = 1,3,5,\cdots \text{ and } 0 \text{ otherwise}$$

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To get a more interesting solution we need to make the x=0 plane hot with V_0 ne 0. We better have plenty of electrician's tape available to insulate the hot x=0 plate from the grounded y=0 and y=a planes. In this case we need to find a set of C_n such that V_0= sum_n=1,2,3,… C_n sin(n pi y /a) . This looks a bit daunting since its hard to imagine a sum of sin functions that sum to a constant. Fortunately Joseph Fourier (1768-1830) discovered a way of matching any reasonable function over an interval with a sum of sinusoidal functions. It is based on the observation that the integral of sin(n pi y/a) times sin(n' pi y/a) over the interval from 0 to a vanishes unless n and n' whenever n and n' are different integers. When they are the same integer the integral becomes a/2. A convenient way of writing this result is in terms of the Kronecker notation: delta_{n n'} = 0 if n ne n' and 1 of n=n'. One way to remember this integral is that < sin^2(n pi y/a) > = ½ the average is the 1/a times the integral and hence the integral is a/2 if n=n' or 0 if it isn't since the average of a sine function product of different frequencies is 0. This allows one to expand any V(0,y) function and find the C_n coefficients as an integral_0^a V(0,y) times sin(n pi y/a) . These integrals are easy to perform for a V(0,y) =V_0 and we thus know C_n for this BC. It is very important that Fourier's trick is not limited to sin and cosine function. In fact some version of the integral vanishing unless n=n' holds for nearly all separation of variable solutions to physically important differential equations. We say that the sinusoidal solutions to the Laplace Eq. form an "orthonormal" set. The fact that the solutions to Schrodinger's Eq. form orthonormal sets is the basis of quantum measurement theory which I hope you will all learn about soon. QM is the jewel of an undergraduate program in physics.

Displaying results

Rescaled for clarity

$x/a =$ — **0.01** — **0.05** — **0.01** — **1**

— 0.01 — 0.05 — 0.01 — 1 - - - SIN(pi x/a)

V/V0 vs y/a (left plot)

V/V0 vs y/a (right plot, rescaled)

$V(x,y)=\dfrac{4V_0}{\pi}\displaystyle\sum_{n=1,3,5\cdots}\dfrac{\exp(-n\pi x/a)\sin(n\pi y/a)}{n}$

Interestingly enough this series can be summed analytically

$V(x,y)=\dfrac{2V_0}{\pi}\tan^{-1}\left(\dfrac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right)$

Large n terms square off $V(x,y)$:

For $x \gg a$ the $\exp\left(\dfrac{-n\pi x}{a}\right)$ kills

all of the terms beyond $n=1$ and

$V(x,y)\rightarrow\dfrac{4V_0}{\pi}\exp\left(-\dfrac{\pi x}{a}\right)\sin\left(\dfrac{\pi y}{a}\right)$

The potential "forgets" the rectangular BC by y $\approx a$ since exp($-\pi$) $=0.043$

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Putting our C_n integrals together with our exponential times sinusoidal separation solutions we have the form for V(x,y) in our domain x>0 , 0 < y < a. The good news is its easy to write. The bad news is it is an infinite series. This would present a big headache to an physicist of the 19th century but is only minor inconvenience in the age of computers. A laptop would barely break out a sweat. If we stuck with just a n = 1, 3 and 5, our answer would wiggle around the real answer where we considered say 100 terms. So I could avoid taking an hour off and avoid writing a program, I relied on the fact that this particular infinite series can actually be summed analytically. You get this goofy looking form in terms of the arc tangent. I plotted V/V0 versus y/a for several slices in x/a. At low values of x (I can't take x=0 since the formula explodes), the solution tries to match the V=V0 BC as best it can while still dropping to zero at x=0 and x=1. As one moves away from the x=0 plane, the V drops off very quickly due to the exp(-n pi x/a) piece of the solution. More interestingly the exp(-n pi x/a) very rapidly kills off all n terms beyond n=1 as x moves away from 0. By the time x reaches "a", our potential has essentially "forgotten" the BC at x =0 The exponential function kills off all of the higher Fourier wiggles which remembered the actual BC. I over plot the actual over the 1st term only in the right plot which has been rescaled to show the shape better. Its easy to see that no matter how we contort the V(0,y) BC we will rapidly just get the V(x,y) = sin(pi y/a) form once x approaches a. We will revisit the forgetful potential again in spherical coordinates.

Laplace Solutions in Spherical Coordinates

In a field free region we have:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Consider solutions of form $V=R(r)\Theta(\theta)$ --w/ no ϕ depend

$$\Theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right)=-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\ell(\ell+1)$$

Interestingly enough the periodicity of θ implies an integral variable of separation $\ell=0,1,2,\cdots$. We thus have: $\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right)=\ell(\ell+1)R$ and

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\ell(\ell+1)\sin\theta\,\Theta$$

The R(r) solution is a power law of the form $R=r^n$

$$\frac{d}{dr}\left(r^2\frac{\partial r^n}{\partial r}\right)=\ell(\ell+1)r^n\rightarrow n(n+1)=\ell(\ell+1)$$

With solutions $n=\ell$ and $n=-(\ell+1)$. Try out $n=-(\ell+1)$

$$-(\ell+1)(-(\ell+1)+1)=\ell(\ell+1)\rightarrow \boxed{R_\ell(r)=A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}}$$

The Θ solutions to:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\ell(\ell+1)\sin\theta\,\Theta$$

are the Legendre polynomials $P_\ell(\cos\theta)$ where $x=\cos\theta$:

$$P_0(x)=1\;;\;P_1(x)=x\;;\;P_2(x)=\frac{3x^2-1}{2};\;P_3(x)=\frac{5x^3-3x}{2}$$

Try out $P_1=\cos\theta$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\cos\theta}{d\theta}\right)=-\ell(\ell+1)\sin\theta\cos\theta=-1(2)\sin\theta\cos\theta$$
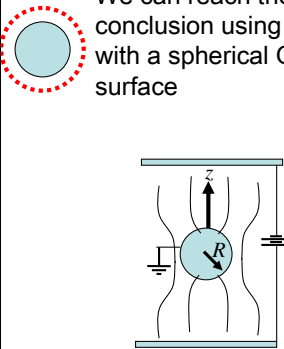
$$-\frac{d\sin^2\theta}{d\theta}=-\sin\theta\cos\theta=-2\sin\theta\cos\theta.\;\text{It checks!}$$

$$\rightarrow \boxed{V=\sum_{\ell=0,1,2}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}\;\text{A very simple form.}$$

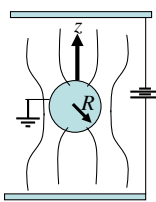A_ℓ and B_ℓ determined by matching boundary conditions.

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We turn next to the separation of variable technique to the Laplace Eq. in spherical coordinates. We note that the Laplacean has a more complicated appearance than the simple form in Cartesian coordinates. This reflects the fact that we don't have constant unit vectors in curvilinear coordinates. Usually we will find a way to orientate the z-axis so our solutions will have no –phi dependence which simplifies things slightly. As before we write a factorizable potential, V is a product of R(r) and Theta(theta). We do the same trick of dividing both sides by V so we have a separate differential eq. in R and Theta. Interestingly enough our constant of separation is an integer L(L+1). I think this is probably because theta is essentially a periodic variable much like the sine of an angle. We now have two differential equations: one for r and one for theta. Although it isn't obvious, a power law for R(r) will work for the r equation. If we plug R(r) = r^n into the R separation equation we get two solutions n = L and n=-(L+1). We write R(r) as a linear as a combination of these two powers with separate constants to match BC. We label the R for this value of the separation constant by the L subscript R_L. Incidentally this is the same L that appears in atomic orbitals in QM or chemistry. The separation solutions for theta are also fairly simple. The are a set of polynomials in cos (theta) called the Legendre polynomials P_L [cos(theta)]. I list a few of them just to show you they are simple. We put the two separation solutions together and have a fairly simple sum involving r^L and 1/r^(L+1) times P_L. The trick in generating a solution for a potential is to find the L's required to match the problem and adjust A_L and B_L to match the BC. Although the result is more complicated, it is often much more useful since one frequently gets an exact solution with just a few terms rather than an infinite series as we will see.

For example if V is spherically symmetric (ie only depends on r) we only have $\ell=0$ and $V = A_0 + \dfrac{B_0}{r}$

$\rightarrow$ All spherically symmetric charge distributions have a Coulomb potential in charge free regions.

We can reach the same conclusion using Gauss's law with a spherical Gaussian surface

A classic problem on $\boxed{V = \sum\limits_{\ell=0,1,2}\left(A_\ell r^\ell + \dfrac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$

is a grounded shell in a uniform electric field $\vec{E}=E_0\hat{z}$

Induced charges on the shell surface will distort the $\vec{E}$

close to the sphere so we expect $\vec{E}(r\rightarrow\infty)=E_0\hat{z}$

On the spherical shell $V(R) = 0$ thus

$A_\ell R^\ell + \dfrac{B_\ell}{R^{\ell+1}} = 0 \rightarrow B_\ell = -A_\ell R^{2\ell+1}$

$V = \sum\limits_{\ell=0,1,2} A_\ell\left(r^\ell - \dfrac{R^{2\ell+1}}{r^{\ell+1}}\right)P_\ell(\cos\theta)$ for grounded shell

Since $E_z = -\dfrac{\partial}{\partial z}V$ we expect $V \rightarrow -E_0 z = -E_0\, r\cos\theta$

The only term which matches is $\ell=1$ since $P_1 = \cos\theta$

Thus $V = -E_0\left(r - \dfrac{R^3}{r^2}\right)\cos\theta$

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Here is a simple example. What if we had a spherically symmetric charge distribution? We know that there can be no theta dependence for the potential. This means we can only have the 0 order Legendre polynomial so L **must** be 0. We have thus found that all spherically symmetric solutions have a potential of the form V(r) = A + B/r and of course the constant A is irrelevant to the fields. This seems like a pretty cool result except it is nothing new. We can get the same result for the E-field from a spherically symmetric charge distribution using Gauss's law with a spherical Gaussian surface.

Here is a more complicated problem. Lets place a grounded metal in an external, constant electrical field. Close to the sphere, there will be surface charges which will distort the E-field but basically the sphere's electric field will die off a long distance from the sphere's center leaving the constant electrical field – lets give it a strength of E_0. Clearly the direction of E_0 will break the spherical symmetry. We will probably get a phi dependent potential (which breaks our method) unless we align our z-axis in the field direction. Hence at long distances -partial V/partial z = E_z = E_0. This implies V(r approaches infinity) = -E_0 z = -E_0 r cos(theta) where we wrote z in spherical coordinates. We can eliminate the B coefficients by the BC that the potential must disappear inside the sphere and thus (by continuity) must disappear on the surface of the sphere at r=R. This leaves us with an sum over possible A__L P_L contributions. One of the term matches all of the BC's. To match the cos(theta) dependence we need a solution of the form V= (A r + B/r^2) cos(theta) plus possible additional L terms. This form goes to A r cos(theta) at large r which must equal -E_o r cos(theta) so clearly A = -E_0 . Fortunately the L=1 term also matches the required power r^1 so we have a winner! We now have the complete solution.

Grounded Sphere in E-field

Writing $E_0 \cos\theta = \dfrac{\tilde{\sigma}_1 \cos\theta}{3\varepsilon_0}$ from

our induced charge expression, we then expect

$V = \dfrac{\tilde{\sigma}_1 \cos\theta R^3}{3\varepsilon_0 r^2}$ for the case where we "glue" a charge

distribution of the form $\sigma = \tilde{\sigma}_1 \cos\theta$ on an insulating

sphere of radius R without an external $\vec{E}$.

It is useful to interpret $V = -E_0\left(r - \dfrac{R^3}{r^2}\right)\cos\theta$

The first term: $V_{1st} = -E_0\, r \cos\theta = -E_0\, z$ must be due

to the external $\vec{E}$. Hence the induced charges create

$V_{2nd} = \dfrac{E_0 R^3}{r^2}\cos\theta.$

We can compute the induced charge on the shell from

$\sigma = \varepsilon_0 E_r = -\varepsilon_0\left(\nabla V\right)_r = -\varepsilon_0 \dfrac{\partial}{\partial r}\left[-E_0\left(r - \dfrac{R^3}{r^2}\right)\cos\theta\right]_R$

$\sigma = \varepsilon_0 E_0\left(1 + 2\dfrac{R^3}{r^3}\right)_R \cos\theta = 3\varepsilon_0 E_0 \cos\theta$

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Lets explore our V = (-E_0 r + B/r^2)cos(theta) solution . We note that E_phi = -(1/r) partial V/partial theta vanishes on the surface of the sphere as it must so the E field is transverse to the spherical surface. Is this solution unique? Could we add some other L terms?? If we think of the sphere as the surface that "bounds" the exterior region, the answer must be no by the Uniqueness Theorem. A less esoteric reason is that if we included another L piece it would require a different Legendre polynomial with a different theta dependence than cos(theta). Hence we could never match a BC that is independent of theta since we are "stuck" with the cos(theta) dependence from the external field. It is easy to compute the surface charge that must be present on the grounded sphere. The surface charge is just epsilon_0 E_r (R) and we can find E_r from the r-hat piece of the negative gradient in spherical coordinates which is just –partial V/partial r. After taking this derivative and going to R+, we obtain a simple surface charge which is proportional to cos(theta). One could integrate the surface charge sigma and find that the total charge on the surface (or anywhere else since all charge must be on the surface) is zero. It is also interesting to interpret our V(r) expression using superposition. Since the first term V_1 = -E_0 r cos(theta) is due to the external field. This suggests that the second term V_2 =+E_0 (R^3/r^2) cos(theta) must be due to the sphere's surface charge. Presumably if we were to glue the bound charge to the sphere so it cannot move and then cut the battery wire and remove the external field we would still have the V_2 =+E_0 (R^3/r^2) cos(theta) or writing this in terms of the surface charge: V_2 = sigma(theta) (R^3/3 epsilon_0) /r^2. We see that the potential due to the glued charge distribution falls off faster than the Coulomb potential. This is because there is no net charge on the sphere. If there was a net charge (say Q) on the sphere we would have a Coulomb potential piece to the potential which falls more slowly (as 1/r) with increasing distance.

Glued charges on a insulated shell

$$V = \sum_{\ell=0,1,2} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Assume a surface charge of the form

$\sigma_\ell(\theta) = \tilde{\sigma}_\ell P_\ell(\cos\theta) \rightarrow$ a single ℓ.

This σ_ℓ is glued on a shell of radius R.

Hence $A_\ell r^\ell + \dfrac{B_\ell}{r^{\ell+1}} = V(r)$. We have 4 BC

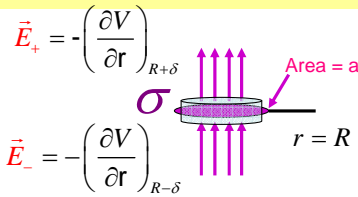(1) For $r < R$, $B_\ell = 0$ or $V \rightarrow \infty$ at $r = 0$

(2) For $r > R$, $A_\ell = 0$ or $V \rightarrow \infty$ at $r = \infty$

(3) $V(R-\delta) = V(R+\delta)$ for infinitesimal δ

(4) $\left(\dfrac{\partial V}{\partial \text{r}} \right)_{R+\delta} - \left(\dfrac{\partial V}{\partial r} \right)_{R-\delta} = -\dfrac{\sigma}{\varepsilon_0}$

The figure illustrates a Gauss's Law argument for the $\partial V/\partial r$ discontinuity due to the surface charge.

$\vec{E}_+ = -\left(\dfrac{\partial V}{\partial \text{r}} \right)_{R+\delta}$

$\vec{E}_- = -\left(\dfrac{\partial V}{\partial \text{r}} \right)_{R-\delta}$

Area = a

σ $r = R$

$\sigma a = \varepsilon_0 \oint \vec{E} \cdot d\vec{a} = a\varepsilon_0 \left[E_+ - E_- \right] =$

$a\varepsilon_0 \left\{ -\left(\dfrac{\partial V}{\partial \text{r}} \right)_{R+\delta} - \left[-\left(\dfrac{\partial V}{\partial \text{r}} \right)_{R-\delta} \right] \right\}$

$\left(\dfrac{\partial V}{\partial \text{r}} \right)_{R-\delta} - \left(\dfrac{\partial V}{\partial \text{r}} \right)_{R+\delta} = \dfrac{\sigma}{\varepsilon_0}$

Apply $V(R-\delta) = V(R+\delta)$ BC

$\left(A_\ell R^\ell \right)_{R-\delta} = \left(\dfrac{B_\ell}{R^{\ell+1}} \right)_{R+\delta} \rightarrow B_\ell = A_\ell R^{(2\ell+1)}$

$\therefore V(r < R) = A_\ell r^\ell$ & $V(r > R) = A_\ell \dfrac{R^{(2\ell+1)}}{r^{\ell+1}}$

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It is interesting to extend our glued charge example **outside** of an arbitrary "glued charge" distribution. To simplify our thinking we will think of our sphere as insulated although the result will also work for a conducting sphere. We begin with our general separation of variable solution. Our strategy is to assume that components of the sigma distribution have a cos(theta) dependence that is proportional to the Legendre polynomial P_L(cos theta). We will show you how to find the sigma_L components shortly. Interestingly enough we will need to consider separate solutions both inside and outside the sphere's surface. Both will be given by our general form but the A, and B coefficients will be different for the two cases. We begin with some general arguments. (1) For the solutions inside of the sphere we need to set all B_L =0 since otherwise V will go to infinity as r approaches 0. (2) Conversely the A_L coefficients must vanish for r > R or the potential will go to infinity as r approaches infinity. Hence for a given sigma_L we have V(r<R) = A r^L P_L(cos theta) and V(r>R) = B P_L(cos theta)/r^(L+1). We want a continuous V so that the two solutions match up at the boundary: V(R-) = V(R+). Hence we can eliminate either A or B since

A R^L = B/R^(L+1) which implies B = AR^(2L+1). But what sets the scale of our remaining constant A? Can't we just set A = 0 and be done with it? Of course this makes no sense since we know V must depend on sigma. Evidently there must be another BC which precludes a A=0 implies V=0 solution. This fourth BC is a discontinuity in partial V/partial r due to the surface charge. The reason for this slope discontinuity is because just below and just above the sphere surface, the surface charge looks like an infinite charged plane. We draw a squat pill-box Gaussian surface whose flux will be (E+ - E-)A where E+/- is the electrical field just above or below the charge at radius R. The normal component of the E-field is – partial V/ partial r ; hence the difference of these derivatives from just above to just below the surface must be sigma/epsilon_0. We can't set A to zero unless there is no charge density glued to the surface and A will be proportional to sigma.

Arbitrary "Glued" charge

Now apply $\partial V/\partial r$ discontinuity:

$$\left(\frac{\partial V}{\partial r}\right)_{R-\delta} = \ell A_\ell R^{\ell-1} P_\ell \;;\; \left(\frac{\partial V}{\partial r}\right)_{R+\delta} = -(\ell+1)A_\ell \frac{R^{(2\ell+1)}}{R^{\ell+2}} P_\ell$$

$$\left(\frac{\partial V}{\partial r}\right)_{R-\delta} - \left(\frac{\partial V}{\partial r}\right)_{R+\delta} = \frac{\tilde{\sigma}_\ell P_\ell}{\varepsilon_0}$$

$$A_\ell \left[\ell + (\ell+1)\right] R^{\ell-1} = \frac{\tilde{\sigma}_\ell}{\varepsilon_0} \rightarrow A_\ell = \frac{\tilde{\sigma}_\ell R^{1-\ell}}{\varepsilon_0 (2\ell+1)}$$

$$\boxed{V(r<R) = A_\ell r^\ell = \frac{\tilde{\sigma}_\ell P_\ell(\theta)}{\varepsilon_0 (2\ell+1)} R\left(\frac{r}{R}\right)^\ell}$$

$$\boxed{V(r>R) = \frac{A_\ell P_\ell R^{(2\ell+1)}}{r^{\ell+1}} = \frac{\tilde{\sigma}_\ell P_\ell(\theta)}{\varepsilon_0 (2\ell+1)} R\left(\frac{R}{r}\right)^{\ell+1}}$$

Check the $\ell=0$ where $\sigma_\ell = \tilde{\sigma}_0 P_0(\cos\theta)$

$$\ell=0 \rightarrow V(r>R) = \frac{\sigma_0}{\varepsilon_0}\frac{R^2}{r^1} = \frac{4\pi R^2 \tilde{\sigma}_0}{4\pi\varepsilon_0 r} = \frac{q}{4\pi\varepsilon_0 r}$$

$$V(r<R) = \frac{\tilde{\sigma}_0}{\varepsilon_0} R = \frac{q}{4\pi\varepsilon_0 R} \text{ (both check)}$$

Now check : $\sigma(\theta) = \sigma_1(\theta) \propto \cos\theta$

$$\ell=1 \rightarrow V(r>R) = \frac{\tilde{\sigma}_1 \cos\theta}{3\varepsilon_0}\frac{R^3}{r^2} \text{ which agrees}$$

with earlier results. $V(r<R) = \dfrac{\tilde{\sigma}_1 \cos\theta\, r}{3\varepsilon_0}$

$$V(r>R) = \frac{\tilde{\sigma}_1 z}{3\varepsilon_0} \rightarrow \vec{E} = -\hat{z}\frac{\partial V}{\partial z} = -\hat{z}\frac{\tilde{\sigma}_1}{3\varepsilon_0} \text{ (Constant!)}$$

Dealing with arbitrary $\sigma(\theta)$

Goal is to write $\sigma(\theta) = \sum_\ell \sigma_\ell(\theta)$.

using fact that $P_\ell(\cos\theta)$ are "orthogonal":

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$\int_{0}^{\pi} P_\ell(\theta) P_{\ell'}(\theta)\sin\theta\, d\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

Hence $\tilde{\sigma}_\ell = \dfrac{2\ell+1}{2}\displaystyle\int_0^\pi \sigma(\theta) P_\ell(\theta)\sin\theta\, d\theta$

Then $V(r>R) = \sum_\ell \dfrac{\tilde{\sigma}_\ell P_\ell(\theta)}{\varepsilon_0(2\ell+1)}\dfrac{R^{\ell+2}}{r^{\ell+1}}$

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We next invoke the slope discontinuity condition at r =R to find A_L for the V(r > R) solution. We have two quick checks of our expression. The first of these is to consider a uniform sigma. Since sigma_L = P_L(cos theta) and P_0 = 1 we know only L=0 contributes. We insert L=0 and sigma_0 into our potential expression and obtain V(r > R) = sigma R^2/(epsilon_0 r) if we multiply the numerator and denominator by 4 pi and write q = 4 pi R^2 sigma. We are left with the Coulomb potential in its standard form. We can do the same for r<R and get V= q/(4 pi epsilon_0 R) which is also correct for a shell potential referenced to infinity. Every thing checks. We can also check our expression for the sigma = sigma_1 cos(theta) case that we considered on our grounded sphere slide. Here P_1(cos theta) = cos (theta) so we know this must correspond to L=1. We plug L=1 into our general expression and get the same form for the potential we discussed before. Another check! The r<R case is particularly interesting since we get a potential which implies a constant electrical field within the sphere. The form is very reminiscent of the E-field for two overlapping spheres worked out in homework with rho d => – sigma_1 . How can we find the sigma_L components for an arbitrary charge distribution. We basically recycle the Fourier trick but this time for Legendre polynomials. Just like the integral of two sine functions vanish when n ne n' , the integral of two Legendre polynomials vanish unless L = L'. They are orthonormal functions. I show you the general form for the integral between two Legendre polynomials. We can write this as an integral over sin(theta) d theta or d cos(theta). The basic idea is say we write sigma(theta) as a sum over sigma_L (each of which is proportional to P_L). If we form the integral Int_-1^+1 Sigma (theta) d cos(theta) only the sigma_L will be left. Once we find all of the sigma_L we can use our V(r) expression which is a sum over sigma_L. A quick check of our results is to go to the case of sigma = Q/(4 pi R) which implies L=0 and we do indeed recover the Coulomb potential.

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Arbitrary glued charge continues…

Example: $\sigma(\theta) = k\cos^2\theta$

$k\cos^2\theta = \sigma_0 + \sigma_1 + \sigma_2$ will be enough

$\tilde{\sigma}_\ell = \dfrac{2\ell+1}{2}\int_0^\pi \sigma(\theta)P_\ell(\theta)\sin\theta\,d\theta$

or $\tilde{\sigma}_\ell = \dfrac{2\ell+1}{2}\int_{-1}^1 \sigma(x)P_\ell(x)\,dx$

where $P_0(x)=1$; $P_1(x)=x$; $P_2(x)=\dfrac{3x^2-1}{2}$

$\tilde{\sigma}_0 = \dfrac{1}{2}\int_{-1}^1 kx^2\,dx = \dfrac{k}{3}$; $\tilde{\sigma}_1 = \dfrac{3}{2}\int_{-1}^1 kx^2\times x\,dx = 0$

$\tilde{\sigma}_2 = \dfrac{5}{2}\int_{-1}^1 kx^2\times\left(\dfrac{3x^2-1}{2}\right)dx = \dfrac{2k}{3}$

Lets check these σ_ℓ

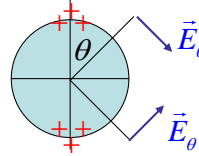$\sigma_0 + \sigma_1 + \sigma_2 = \dfrac{k}{3} + 0 + \dfrac{2k}{3}\left(\dfrac{3x^2-1}{2}\right) = kx^2$

$V(r>R) = \sum_\ell \dfrac{\sigma_\ell(\theta)}{\varepsilon_0(2\ell+1)}\dfrac{R^{\ell+2}}{r^{\ell+1}} = \dfrac{kR^2}{3\varepsilon_0 r} + \dfrac{2kR^4}{15\varepsilon_0 r^3}\left(\dfrac{3x^2-1}{2}\right)$

$\rightarrow \boxed{V(r>R) = \dfrac{kR^2}{3\varepsilon_0 r} + \dfrac{kR^4}{15\varepsilon_0 r^3}\left(3\cos^2\theta - 1\right)}$
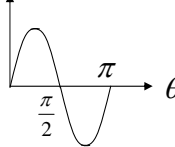
Look at E_θ

$\nabla = \hat{r}\dfrac{\partial}{\partial r} + \dfrac{\hat{\theta}}{r}\dfrac{\partial}{\partial\theta} + \dfrac{\hat{\phi}}{r\sin\theta}\dfrac{\partial}{\partial\phi}$
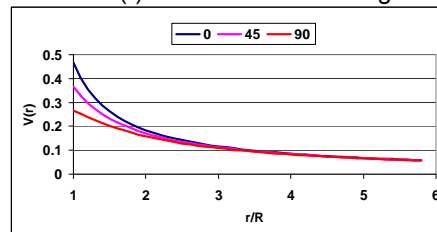
$E_\theta \propto -\dfrac{\hat{\theta}}{r}\dfrac{\partial}{\partial\theta}\dfrac{k\cos^2\theta}{r^3} = -\dfrac{\hat{\theta}}{r^4}2k\cos\theta\sin\theta$

$E_\theta \propto \sin 2\theta$

Plot V(r) for three different angles

V(r) "forgets" the $\cos^2\theta$ beyond r≈3R

Here is a specific example. Consider a glued charge distribution proportional to cos^2 (theta). Since the largest power of cos(theta) in a P_L is cos^L (theta) we know the highest sigma_L will be sigma_2. We get the sigma_L contributions by integrating K cos^2 (theta) with P_L and get non-zero contributions for sigma_0 and sigma_2. We can add our sigma_0 and sigma_2 pieces together to verify that they add up to k cos^2 (theta) which they do. We can find V(r) using our general expression and can find the E fields from the gradient expression in spherical coordinates. As a quick check of the result we look at the theta component of the electric field. From the gradient of our potential we find that E_theta is proportional to sin 2 theta which means it points away from the north and south poles and disappears at the equator. This makes a great deal of sense since positive charge accumulates at the two poles and E will point away from the positive charge at the poles. The lower figure gives another illustration of how solutions to Laplace's equation at large distances can "forget" their short distance boundary conditions. We show the potential at theta = 0 , 45, and 90 degrees as a function of radius. At short distances, we see a significantly smaller potential at the equator compared to the poles. At long distances there is very little dependence on "latitude". This is very easy to understand in the context of the multipole expansion which we address shortly. The first of the two terms in the potential is called the "monopole" term which looks like the Coulomb potential of a single charge and is thus isotropic (eg the same in any direction). The theta dependence which gives a larger potential at the pole compared to the equator comes from the second or "dipole" term. Since the dipole term dies out much faster with distance than the monopole term, the anisotropy dies away with increasing distance from the sphere.
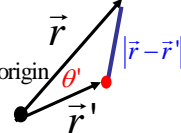
Multipole Expansion

Our glued charge expression

$$V(r > R) = \sum_{\ell} \frac{\tilde{\sigma}_{\ell}(\theta) P_{\ell}(\cos\theta)}{\varepsilon_0 (2\ell+1)} \frac{R^{\ell+2}}{r^{\ell+1}}$$

is an example of multipole expansion

which is useful way of desribing V(r)

at large r based on this generating function

expansion:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0,1,2\cdots} \frac{r'^{\ell}}{r^{\ell+1}} P_{\ell}(\hat{r}\cdot\hat{r}')$$

where $\vec{r}' = r'\hat{r}'$ and $\vec{r} = r\,\hat{r}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

$$V(\vec{r}) = \sum_{\ell=0,1,2\cdots} \frac{\int \rho(\vec{r}') r'^{\ell} P_{\ell}(\hat{r}\cdot\hat{r}') d\tau'}{4\pi\varepsilon_0 r^{\ell+1}}$$

$$V(\vec{r}) = \sum_{\ell=0,1,2\cdots} \frac{m_{\ell}}{4\pi\varepsilon_0 r^{\ell+1}} \text{ where } m_{\ell} = \text{multipole}$$

$$m_{\ell} = \int \rho(\vec{r}') r'^{\ell} P_{\ell}(\hat{r}\cdot\hat{r}') d\tau' \; ; \; \to m_0 = \int \rho(\vec{r}') d\tau' = Q$$

$$m_1 = \int \rho(\vec{r}') r'(\hat{r}\cdot\hat{r}') d\tau' = \hat{r}\cdot\int \rho(\vec{r}') \vec{r}' d\tau' = \hat{r}\cdot\vec{p}$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

The first terms of the glued charge expansion are

$$V(r) = \frac{\tilde{\sigma}_0 R^2}{\varepsilon_0 r} + \frac{\tilde{\sigma}_1 R^3 \cos\theta}{3\varepsilon_0 r^2}. \text{ consistant if}$$

$$\frac{Q}{4\pi} = \tilde{\sigma}_0 R^2 \text{ and } \frac{\hat{r}\cdot\vec{p}}{4\pi} = \frac{\tilde{\sigma}_1 R^3 \cos\theta}{3}$$

The first is obvious $\to Q = 4\pi R^2 \tilde{\sigma}_0$

2nd implies $\vec{p} = \dfrac{4\pi\tilde{\sigma}_1 R^3 \hat{z}}{3}$ since $\hat{r}\cdot\hat{z} = \cos\theta$

$\boxed{\text{Check this}}$ $\vec{p} = \int \rho(\vec{r}') \vec{r}' d\tau' \to \int \sigma(\theta) \vec{r}\, da$

$$\vec{p} = \int_{-1}^{1} R^2 \tilde{\sigma}_1 \cos\theta\, d\cos\theta \int_{0}^{2\pi} d\phi \begin{bmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{bmatrix}$$

$$\vec{p} = \int_{-1}^{1} R^2 \tilde{\sigma}_1 \cos\theta \left(2\pi R \cos\theta\, \hat{z}\right) d\cos\theta = \frac{4\pi\tilde{\sigma}_1 R^3 \hat{z}}{3}$$

The glued charge expansion is a nice introduction to multipole expansions. This is a useful representation for the potential due to a localized charge distribution when one is far away from the charge. Hence it is an expansion in powers of 1/r where r is the distance. The multipole expansion is based on expanding 1/|r – r'| in powers of 1/r and r' (similar to the glued charge). The angular dependence of the potential is written in terms of the Legendre polynomials which depend on cos(theta) where theta is the angle between the source point r' and the observation point r which can be written as r-hat dot r'-hat. This expansion represents one way of defining the Legendre polynomials and is often called the "generating function" of the Legendre polynomials. It is perfect for writing the potential of a charge distribution owing to the superposition form for V(r). Inserting the generating function representation we can write the potential as various integrals over the charge density rho. These integrals are the multipole terms. We next write the first two multipoles: the monopole and the dipole. The dipole is a vector which is a displacement vector times the charge density integrated over the charge volume. The dipole contribution to the potential is the dot product of the unit vector to the observer and the dipole moment p-vec. This contribution adds a potential term proportional to 1/r^2 which falls faster than the monopole term (or total charge) which falls as 1/r like a point charge. Comparing the first two terms of the multipole expansion to the glued charge Legendre expansion, we see we get agreement as long as sigma_0 and sigma_1 are related to Q and p-vec as indicated. The Q relation is simple to see since Q must equal sigma_0 times the area of the sphere which sigma_0 is glued on. The p-vec expression takes a little more work. To get the cos(theta) term of the glued charge expression, we need the dipole moment (p-vec) to lie in the z-hat direction since r-hat dot z-hat is cos (theta). This is because the glued charge distribution was chosen to have no phi dependence and thus only depends on theta. We check this consistency by computing the dipole moment of the sigma_1 charge distribution glued to a ball of radius R. Our first step is to convert the volume integral to a surface integral over the ball in the obvious way. This is an integral over d cos (theta) d phi. We write the r' displacement vector as a column vector involving R, theta and phi. Only the z-hat component survives the phi integral. We are left with a simple integral involving two powers of cos (theta) : one from sigma_1 and one from r'_z. Indeed we recover the form to make things consistent.

The physical dipole

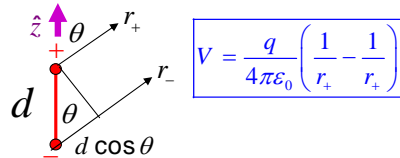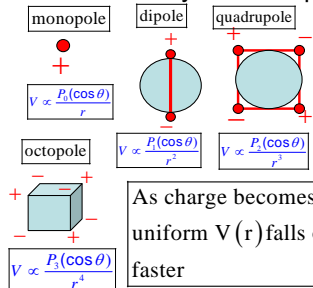$\vec{p} = \int \rho(\vec{r}\,') \, \vec{r}\,' \, d\tau'$ with

$\rho(\vec{r}\,') = q \left[\delta^3(d \, \hat{z}) - \delta^3(\vec{0}) \right]$

$\vec{p} = q \, d \, \hat{z} - q(0) = q \, d \, \hat{z}$

$V = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_+} - \dfrac{1}{r_+} \right)$

Physical multipoles

monopole

$V \propto \dfrac{P_0(\cos\theta)}{r}$

dipole

$V \propto \dfrac{P_1(\cos\theta)}{r^2}$

quadrupole

$V \propto \dfrac{P_2(\cos\theta)}{r^3}$

octopole

$V \propto \dfrac{P_3(\cos\theta)}{r^4}$

As charge becomes less uniform $V(r)$ falls off faster

The physical dipole

$V \to \dfrac{q}{4\pi\varepsilon_0}\left[\dfrac{1}{r} - \dfrac{1}{r + d\cos\theta} \right]$

$V \to \dfrac{q}{4\pi\varepsilon_0 r}\left[1 - \left(1 + \dfrac{d\cos\theta}{r} \right)^{-1} \right]$

$\left(1 + \dfrac{d\cos\theta}{r} \right)^{-1} = 1 - \dfrac{d\cos\theta}{r} + \cdots$

$\to \boxed{V = \dfrac{qd\cos\theta}{4\pi\varepsilon_0 r^2} + \mathcal{O}\left(\dfrac{1}{r^3} \right)}$

$V \approx \dfrac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$ where $\vec{p} = qd \, \hat{z}$

The physical dipole is not ideal since it contains additional $P_l(\cos\theta)$ terms in the V expansion. The glued charge σ_1 however is an ideal dipole.
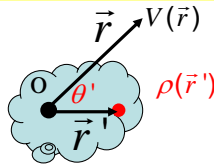
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Our glued charge problem is an example of a multipole expansion. As the charge distributions become more complicated and structured (corresponding to higher Legendre polynomials) their potentials fall faster with increasing radius. Hence a constant surface charge density has a "Coulomb" potential that falls as 1/r. A surface charge density proportional to cos (theta) falls as 1/r^2. A surface charge density proportional to P2(cos theta) falls as 1/r^3, The figure illustrates that each of these potentials that fall faster than 1/r have a net charge of 0 and the more "wiggles" in the charge distribution, the faster its potential falls with increasing distance. We give some examples of "physical" multipoles. A point charge is an example of a monopole whose potential falls as 1/r. Two separated opposite charges is an example of a dipole whose potential falls as 1/r^2. Four alternating charges on a square is a quadrupole whose potential falls as 1/r^3. Finally 8 alternating charges on a cube is an example of an octopole. By physical multipoles we mean they are not the ideal multipoles with potentials that are precisely proportional to a pure Legendre polynomial. For example the physical dipole (charge dumbbell) also contains some L=3 (octopole) and L=5,7,9,… pieces. Similarly the physical quadrupole or square of charges also contains some L=4,6,8… pieces in its potential. The charge dumbbell is an easy to understand example of a physical dipole. We evaluate it using our dipole moment integral over the charge distribution and source vector r'. For fun we write this in terms of two three dimensional delta functions located at the origin (for the negative charge) and at d z-hat for the positive charge. This gives us dipole moment of p-vec equal to q times d along the z-hat direction. As a check, we write its potential as the superposition of the potential of a positive and negative charge.

As shown in the figure, the negative charge is slightly further away from a distant observer by a factor of d cos theta when the dipole is aligned along the z axis. To an observer who is a long distance from the dipole, the vectors from each charge are nearly parallel and hence the distance difference is the adjacent side of a right triangle and is approximately d cos theta. We can expand the 1/[r + d cos(theta)] in the potential expression using the binomial theorem:

(1 + x)^N is approximately 1+N x when x << 1. Here N =-1. Our approximate potential expression is proportional to cos(theta)/r^2 in the large distance limit which means the charge dumbbell has approximately the same potential as an ideal dipole or P_L/r^(L+1) with L=1. We write this potential is in terms of the dipole moment p = q d and get exactly the same p-vec dot r-hat/r^2 form that we had in the multipole expansion that used the generator function.

Origins in Multipole Expansion

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2} \quad \text{where}$$
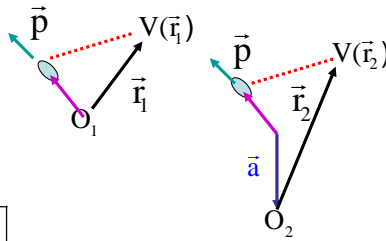
$$Q=\int \rho(\vec{r}\,')\,d\tau' \quad \text{and} \quad \vec{p} = \int \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau'$$

Clearly Q does not depend on the origin O but it looks like $\vec{p}$ (the dipole moment) does because of $\vec{r}\,'$ in integral giving position of $\rho(\vec{r}\,')d\tau'$ relative to O. Lets change O:

$$\vec{r}\,' \to \vec{r}\,'+\vec{a}\;;\; \vec{p} \to \vec{p}_{shift} = \int \rho(\vec{r}\,')(\vec{r}\,'+\vec{a})\,d\tau'$$

$$\vec{p}_{shift} = \vec{p} + \int \rho(\vec{r}\,')\,\vec{a}\,d\tau' = \vec{p} + Q\vec{a}$$

Special case is $Q = 0$ then $\vec{p}_{shift} = \vec{p}$

Although $\vec{p}$ is same under an O shift, shifting O will also change $\vec{r}$ and $\hat{r}$

$$V(\vec{r}_1) = \frac{\hat{r}_1\cdot\vec{p}}{4\pi\varepsilon_0 |\vec{r}_1|^2} \neq \frac{\hat{r}_2\cdot\vec{p}}{4\pi\varepsilon_0 |\vec{r}_2|^2} = V(\vec{r}_2)$$

but relative positions are the same!

A large $\vec{a}$ affects other moments.
Moral: put O close to dipole charges & keep $\vec{a}$ small on size scale of $\rho(\vec{r})$ for good convergence.

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The charge density integrals are essentially the multipole moments. We write out the first two terms explicitly. The monopole term is the integral over the charge distribution which is just the total charge Q. The dipole moment term involves the integral over r' which can be thought of as the charge averaged source vector r'. We essentially got the same expansion for the voltage To what extent do the multipole moments depend on the choice of origin? Clearly the monopole contribution does not since it is just the total contained charge. But it looks like the dipole moment might since it involves the source vector r'. In order to explore this, we consider shifting r' by a vector **a**. This is equivalent to shifting the origin by a vector - **a** .We see the shifted version of the dipole moment is the same as the original version as long as the net charge is zero. This will be the case for electrical fields in matter that we will consider in the next chapter.
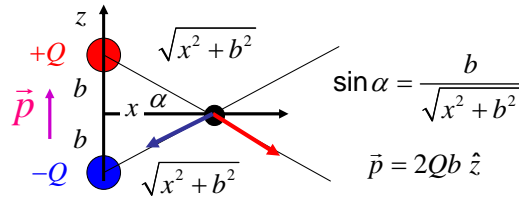
E-field of Dipole

Easy to get $\vec{E}$ for $V(\vec{r}) = \dfrac{\hat{r} \cdot \vec{p}}{4\pi\varepsilon_0 r^2}$

$E(x,y,z) = -\dfrac{1}{4\pi\varepsilon_0} \vec{\nabla} \dfrac{xp_x + yp_y + zp_z}{\left(x^2 + y^2 + z^2\right)^{3/2}}$

$\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}\cdot\hat{r}\right)\hat{r} - \vec{p}\right]$

Compute $\vec{E}$ from Coulomb law for $\vec{r} = x\hat{x}$

$\sin\alpha = \dfrac{b}{\sqrt{x^2 + b^2}}$

$\vec{p} = 2Qb\,\hat{z}$

$E_z = 2 \times \dfrac{-Qb}{4\pi\varepsilon_0 \sqrt{x^2+b^2}^{\,3}}$

$\vec{E}_{\text{dipole form}}(\hat{r} = \hat{x}) \rightarrow \dfrac{-\vec{p}}{4\pi\varepsilon_0 r^3} \text{ since } \vec{p}\cdot\hat{r} = 0 \xrightarrow{\ x \gg b\ } \dfrac{-p_z\hat{z}}{4\pi\varepsilon_0 x^3} = \dfrac{-2qb\hat{z}}{4\pi\varepsilon_0 x^3}$

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We can easily work out the electrical field from the dipole potential. As a sanity check lets evaluate the field for the case of a dipole aligned along the z-axis when our observer is a long distance away on the x-axis using Coulomb's law in order to check our dipole field expression. By drawing the fields from the red and blue charges we see that Ex vanishes and only Ez is present. We can double Ez from either the red or blue charge which depends on 1/(x^2 + b^2) and sin(alpha). We can evaluate sin(alpha) from trig to get our Ez expression from Coulomb's law. We check it against the dipole field formula. Here our observer direction is r-hat = x-hat and thus the 3 (p-hat dot r-hat) r-hat term vanishes. This leaves the term proportional to the dipole vector p. Putting in 2 Q b for p_z we get the same expression in the x >> b limit.