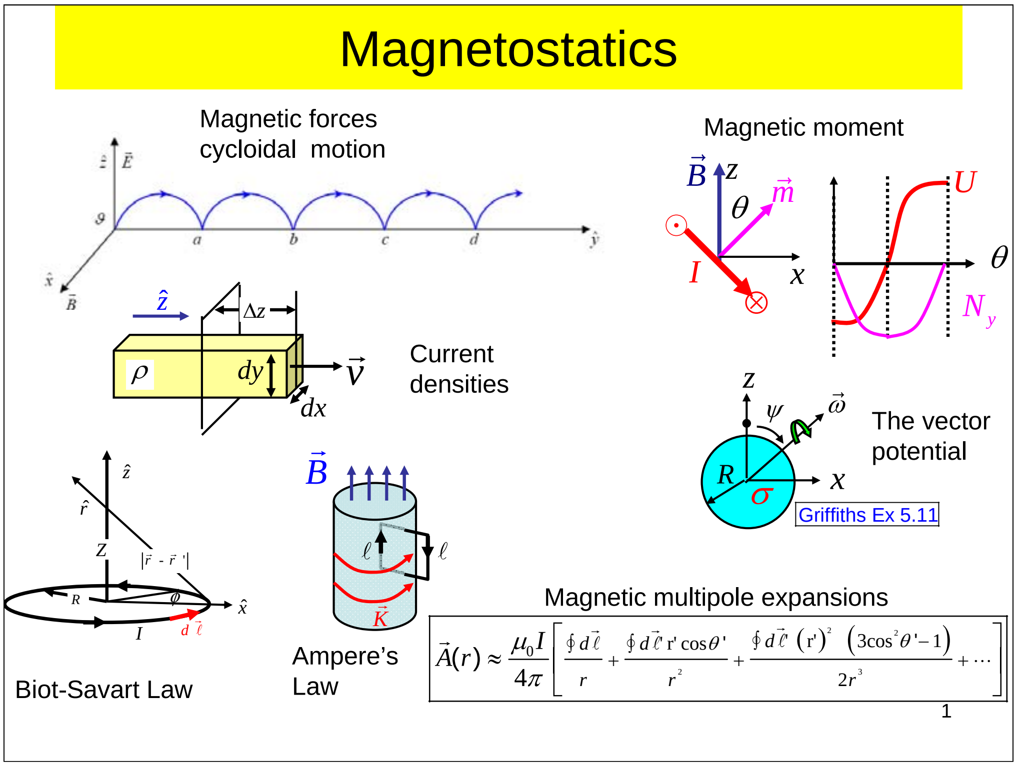


This chapter describes magnetostatics in a vacuum. By magnetostatics we, of course, don't mean that the charges are static but rather the magnetic fields, electric fields and currents are constant in time. We begin with a discussion of magnetic forces and give an example of motion of a charged particle in a crossed electric and magnetic field. Interestingly enough the particle moves in a cycloid orbit. We next discuss the various forms of currents that we will use throughout Physics 435 and 436. These include simple currents (I) measured in Ampere's , surface currents (K) measured in Ampere/meter, and (volume) current densities (J) measured in Ampere/meter^2. We next consider the two methods for calculating the B-field created by currents. The general method (analogous to Coulomb's Law) is called the Biot-Savart Law. For highly symmetric situations one can get the answer much quicker using Ampere's Law (analogous to Gauss's Law) . We turn next to a discussion of the magnetic moment of a current loop and how it affects the torque and potential energy of the loop when placed in a magnetic field. We will discuss the vector potential A(r) from which one can get the magnetic field B through the curl. The vector potential is how magnetic forces are introduced in Lagrangean mechanics and quantum mechanics. Finally, in preparation for our discussion of magnetic fields in materials, we will develop the multipole expansion for the vector potential from currents. You will notice lots of similarities in the A expansion and the V multipole expansion of electrostatics.

Magnetic fields and forces

When both $\vec{E}$ & $\vec{B}$ present we have

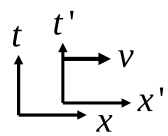
Lorentz force law: $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

In the frame where v=0 we can think of this as:

$$q\vec{E}' = q\vec{E} + q\vec{v} \times \vec{B} \rightarrow \vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

In moving frame $\vec{E}$ and $\vec{B}$ are mixed up.

Field transformation looks similar to time-space mixing in Special Relativity.



$$x = \frac{x' + vt'}{\sqrt{1-(v/c)^2}} \rightarrow x' + vt'$$

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-(v/c)^2}} \rightarrow t' + \frac{v}{c^2}x'$$

$\vec{B}$ for moving charge heightens analogy: $\vec{B}(\vec{r}) = \frac{\vec{v}}{c^2} \times \vec{E}(\vec{r})$

Is $(\vec{E} \quad \vec{B}) \Leftrightarrow (x \quad t)$? Not exactly

but SR does relates them.

We combine $\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}$ with

$$\vec{E}(\vec{r}) = \frac{q(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|^3}$$ to get

$$\vec{B} = \frac{q}{4\pi\varepsilon_0 c^2}\vec{v} \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \Rightarrow \text{Biot-Savart}$$

Valid when $v \ll c$. Recall $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$\frac{1}{\varepsilon_0 c^2} = \mu_0 \rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi\varepsilon} q\vec{v} \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}}$$

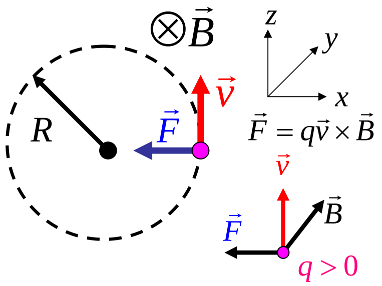
$$\vec{B} = \frac{\mu_0}{4\pi\varepsilon} q\vec{v} \times \frac{\hat{𝓇}}{𝓇^3}\left[\frac{1-\beta^2}{\left(1-|\hat{𝓇} \times \vec{\beta}|^2\right)}\right]_{rel}$$

where $\vec{\beta} = \vec{v}/c$, $\hat{𝓇} = \vec{r} - \vec{v}t$

2

Much later in the course we will show that magnetic and electric fields and forces can be transformed into each other according to special relativity. In fact understanding the deep connection between electricity and magnetism was one of the major motivations for Einstein to invent special relativity. For example we can think of the magnetic force as part of a transformed electric field which looks very similar to the relativistic transformation between time and space. But we will show that fields undergo a tensor transformation rather than the vector transformation used for time-space. If we view the B-field as the Coulomb law E-field transformed in a moving frame we essentially get the Biot-Savart law for a moving charge that shows how a moving charge can create a magnetic field. Our expression is only good at velocities very small compared to the speed of light. Our Biot-Savart expression is in terms of epsilon_0 and the speed of light. In Physics 212 you learned that Maxwell's triumph was the realization that light is an electromagnetic phenomena and that the speed of light can be related to the constants of electrostatic (epsilon_0) and magnetostatics (mu_0) that can be measured or defined in 19th century lab bench experiments. We thus replace our "relativity" expression with the more conventionally written Biot-Savart law in terms of mu_0. It is interesting to note that according to the Biot-Savart law the magnetic field has the same 1/r^2 term that the Coulomb electric field has but it is crossed into the velocity of the charge owing to the relativistic "origin" of the magnetic field. A velocity dependence to the force law is unusual and required some fundamental changes to mechanics as we show later in this Chp. In Physics 436 we will obtain the relativistic correction to the Biot-Savart form for a charge moving with a constant velocity in terms of Griffith's script r notation. More interestingly, we will get the exact Biot-Savart expression by a tensor transformation of the E-field from a long charged cylinder. The moving cylinder acts as a the current without the displacement contribution due to the time varying E-field of the moving point charge.

Magnetic force: Cyclotron motion



Find the period. $\tau = \frac{2\pi R}{v} = \frac{2\pi \frac{mv}{qB}}{v} = \frac{2\pi m}{qB}$

In NR limit τ is independent of v!

$\omega = \frac{2\pi}{\tau} = \frac{qB}{m}$ (cyclotron frequency)

This allows one to make an accelerator with no need for frequency control.

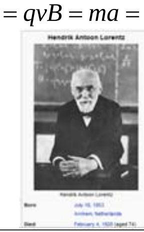
Informal solution

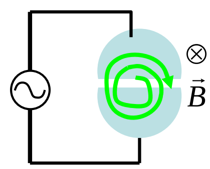
$\vec{F} \perp \vec{B} \perp \vec{v} \rightarrow \vec{F} = q\vec{v} \times \vec{B} = qvB$

Circular motion implies

$a_{cent} = \frac{v^2}{R} \rightarrow \vec{F} = qvB = ma = m\frac{v^2}{R}$

$\rightarrow R = \frac{mv}{qB}$





Run AC at ω_{cyc} .
Acceleration from E-field between Dee's. Focusing from fringe field of magnet.

E. Lawrence (1930)

Unfortunately cyclotron only works for v << c

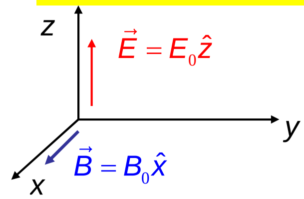
Note: a steady $\vec{B}$ never does work on q.

Power = $\vec{F} \bullet \vec{v} = \left(q\vec{v} \times \vec{B}\right) \bullet \vec{v} = 0$ (Always)

3

We begin by discussing magnetic forces. The magnetic force equation is named in honor of Hendrik Lorentz a contemporary of Einstein although magnetic forces were known in ancient Greece. Our form is completely relativistically correct as we show much later. We begin with the hopefully familiar problem of a charged particle in a circular orbit due to a constant and uniform magnetic field. We assume that the velocity of the charged particle is perpendicular to the magnetic field. In this case the cross product is just the product of the velocity and the field and the force is just qVB. Since Force is proportional to V cross B it is always perpendicular to velocity. As a result the charge travels in a circle with the centripetal acceleration pointing to the center of the circle. As you recall the centripetal acceleration is of the form v^2/R. By setting the magnetic force to the mass times the centripetal acceleration you can easily find the radius of the circular orbit of the charge. The radius is just the momentum of the particle divided by the charge times field. It is also easy to calculate the period of the orbit. This is just the circumference of the orbit divided by the velocity. Interestingly enough the period is independent of velocity. A key parameter is called the cyclotron frequency which depends only on the B-field and the charge to mass ratio of the charged particle. In the 1930's E.O. Lawrence realized that the fact that the orbital frequency was independent of velocity allows for a very simple particle accelerator appropriate to 1930 technology. The idea is to run a pair of "Dee" shaped electrodes at the cyclotron frequency of the particle. The particle gets accelerated when it crosses the gap between the dee's and the field essentially disappears when the particle is within the dee shell. Since the period is independent of velocity the RF frequency can be constant with no need for continuous adjustment. Lawrence was rather lucky since there are some interesting dynamics which keeps the beam focused in the accelerator and keeps the particles in phase with the RF. He was also unlucky. As the particle becomes relativistic the orbit no longer becomes independent of the particle energy and the RF is no longer in phase with the orbit. This sets a limit to the energy that can be reached by a cyclotron. One final point is that static magnetic fields never do work on charges. This is because the power delivered by a force is the dot product of the force and the particle velocity. Since the force is transverse to the velocity the power delivered by the magnetic field is always zero.

Crossed E and B: cycloid motion



$$\vec{v} = (\dot{x} \quad \dot{y} \quad \dot{z}) \quad \vec{a} = (\ddot{x} \quad \ddot{y} \quad \ddot{z})$$

$$\vec{F} = qE_0\hat{z} + q\vec{v} \times (B_0\hat{x}) = m\vec{a}$$

$$m(\ddot{x} \quad \ddot{y} \quad \ddot{z}) = qE_0\hat{z} + q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ B_0 & 0 & 0 \end{vmatrix}$$

$$(\ddot{x} \quad \ddot{y} \quad \ddot{z}) = \frac{q}{m}(0 \quad B_0\dot{z} \quad E_0 - B_0\dot{y})$$

Let $\omega = \dfrac{qB_o}{m}$ & $V = \dfrac{E_0}{B_0}$

Then Eq. of Motion is

$$(\ddot{x} \quad \ddot{y} \quad \ddot{z}) = \omega(0 \quad \dot{z} \quad V - \dot{y})$$

Solution for initial conditions $\vec{r}(0) = 0$ & $\vec{v}(0) = 0$:

$$\vec{r} = \left(0 \quad \frac{V}{\omega}(\omega t - \sin\omega t) \quad \frac{V}{\omega}(1 - \cos\omega t)\right)$$

Easy to see that solution has $\vec{r}(0) = 0$

For further checks we need

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \text{ and } \vec{a} = \ddot{\vec{r}}$$

$$\vec{v} = (0 \quad V(1 - \cos\omega t) \quad V\sin\omega t)$$
$$\vec{a} = (0 \quad V\omega\sin\omega t \quad V\omega\cos\omega t)$$

Solution also has $\vec{v}(0) = 0$

Finally check the Eq. of Motion

$\ddot{y} = \omega\dot{z}$ and $\ddot{z} = \omega(V - \dot{y})$

$$[V\omega\sin\omega t]_{\ddot{y}} = \omega[V\sin\omega t]_{\dot{z}}$$

$$[V\omega\cos\omega t]_{\ddot{z}} = \omega\left\{V - [V(1 - \cos\omega t)]_{\dot{y}}\right\}$$

4

Griffiths Ex. 5.2 discusses a particularly elegant example of magnetic and electric force laws by describing the motion of a charge in crossed electric and magnetic fields. We use Newton's dot notation for time derivative throughout. The equation of motion is fairly simple since these are static, uniform fields. We can simplify the equation even further by using the cyclotron frequency to parameterize B and the "selection velocity" V to parameterize E/B. We note that the form of the Lorentz force means that vB must have the same units of E so E/B indeed has the units of a velocity. We then have a very simple Eq. of motion which relates the acceleration to the velocity where the cyclotron frequency gives the extra "per second". Although this requires a solution of coupled differential equations, the solution is rather straightforward and involves sinusoidal solutions. To proceed we need initial conditions on position and velocity. We start the charge at the origin with zero velocity and show the solution with these initial conditions. You can easily see all three components of the displacement vector disappear at t=0. Differentiating the displacement gives the velocity which also disappears at t=0. We differentiate the velocity to get the acceleration components. This allows us to check the equation of motion in the y and z direction. The subscripts on the last boxed equations interpret the bracketed quantities in terms of the components of velocity and acceleration.

The cycloid continues…

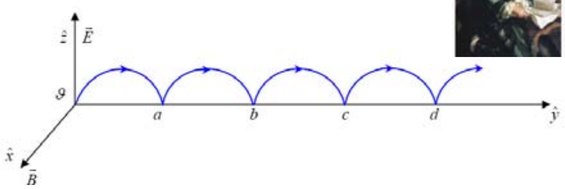
To understand solution let $R = \frac{V}{\omega} = \frac{mV}{qB_0}$

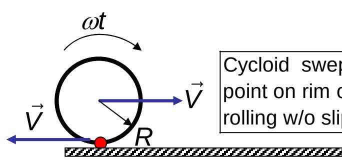
$$\vec{r} = \left(0 \quad \frac{V}{\omega}(\omega t - \sin\omega t) \quad \frac{V}{\omega}(1 - \cos\omega t)\right)$$

$y = R(\omega t - \sin\omega t)$; $z = R(1 - \cos\omega t)$

Using $1 = \sin^2\omega t + \cos^2\omega t$ we see:

$(y - Vt)^2 + (z - R)^2 = R^2$





Cycloid swept out by point on rim of wheel rolling w/o slipping

Fun facts about variables

$\omega = \frac{qB_0}{m}$ = cyclotron freq

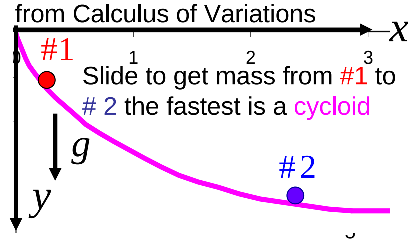
$V = \frac{E_0}{B_0}$ is undeflected velocity

$F_{tot} = qE_0 - qVB_0 = 0 \rightarrow V = E_0 / B_0$

$R = \frac{V}{\omega} = \frac{mV}{qB_0}$ is circle radius if $v = V$

Fun facts about cycloids

Recall Brachistochrone problem from Calculus of Variations



Slide to get mass from #1 to # 2 the fastest is a cycloid

The shape of the cross field orbit is fairly interesting in its own right. If we plot the displacement as a function of time, the orbit sweeps out a cycloid. We can think of a cycloid as a “rolling circle” or the path that a spot on the rim of a wheel sweeps out when it rolls without slipping. To see this we rewrite the solution in terms of a radius R which is the radius that a charge would have if it had velocity equal to the “selection velocity” V. We write the time dependence of y and z in these variables and use cos^2 + sin^2 to eliminate the trig functions. We essentially have the equation of a circle where z is centered on R and y is centered on a point which moves with the “selection velocity”. We note the sharp cusps that occur at the bottom where z=0. These are fairly easy to understand. For the rolling wheel a point on the rim is instantaneously at rest. This means the magnetic force vanishes and there is no acceleration in the y (horizontal) direction. There is however an acceleration in the vertical (z direction) since the E-field still provides a force. Hence the charge rapidly picks up a z velocity due to the E-field acceleration, and much more slowly pick up a horizontal velocity since both v and a are zero in the y-direction. The variables I am using have simple physical significance. Omega is the cyclotron frequency. The “selection velocity” is the velocity that a charge will have so that the Lorentz magnetic force is just canceled by the electric force for crossed fields. This principle can be used to select out specific ion velocities for experiments. Finally R is the magnetic radius for a charge traveling at the selection velocity. The cycloid played a very interesting role in the history of mathematics as one of the first uses of the calculus of variations. The idea was to find the shape of a slide that allows a mass to slide without friction between two points in the shortest amount of time. Euler was able to prove the fastest slide was in the shape of a cycloid. You can easily imagine that a optimal shape exists since you get lots of speed in a steep slide but cover little horizontal distance, and vice versa. It took a genius like Euler to construct the mathematics to find the optimum shape in the 1700’s

Current form of Lorentz & Biot-Savart

For $\{q_i\}$ we use superposition to write Lorentz and Biot-Savart

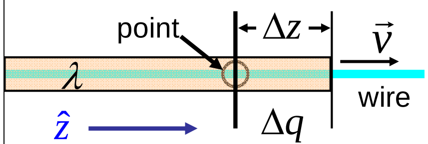
$\vec{F} = q\vec{v}\times\vec{B} \;\rightarrow \vec{F} = \sum_i q_i v\left(\times\vec{B}\right)$. Similarly

$$\vec{B} = \sum_i q_i\vec{v}\left(\times\frac{\mu_0}{4\pi}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right) = \sum_i q_i\vec{v}\left(\times\frac{\mu_0}{4\pi}\frac{\vec{\mathfrak{r}}}{\mathfrak{r}^3}\right)$$

Lorentz and B-S have form: $\sum_i q_i\vec{v}(\;)$

Simplest case uses $\lambda = \dfrac{dq}{dz}$

$dq = \lambda dz \;;\; \sum_i q_i\vec{v}(\;)\rightarrow\int\vec{v}\lambda dz(\;) = \int v\hat{z}\lambda dz(\;)$



We can show $I = \lambda v$ where $I = \dfrac{\Delta q}{\Delta t}$

Here Δq is charge past point in Δt

$\Delta q = \lambda\,\Delta z \;;\; \Delta z = v\Delta t \rightarrow \Delta q = \lambda\, v\Delta t \rightarrow I = \lambda v$

$\sum_i q_i\vec{v}(\;)\rightarrow\int I\,\hat{z}dz(\;)$ in general $\hat{z}dz = d\vec{\ell}$

$$\boxed{\sum_i q_i\vec{v}(\;)\rightarrow\int I\,d\vec{\ell}(\;)}$$

thus $\vec{F} = I\int d\vec{\ell}\times\vec{B}$ and $\vec{B}(\vec{r}) = \dfrac{\mu_0 I}{4\pi}\int\dfrac{d\vec{\ell}\times\vec{\mathfrak{r}}}{\mathfrak{r}^3}$

We also have a volume integral form:

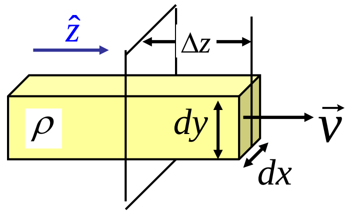
$\sum_i q_i\vec{v}(\;)\rightarrow\int_V\vec{v}\rho d\tau(\;) = \int_V\vec{J}d\tau(\;)$ where $\vec{J} = \rho\vec{v}$

We will show $\int_S\vec{J}\bullet d\vec{a} = I$

6

Rather than dealing with individual charges, we cast the Lorentz force law and the Biot-Savart law in current form. Our first step is to assume superposition and assume the force on a wire carrying a collection of charges is just the sum of the forces on each charge. Similarly the field created by a collection of charges in a wire is the sum of the fields created by each charge. Both the force and field expressions are sums over charge times velocity crossed into another vector. For the force this is the field, for the Biot-Savart field this is the r-hat/r^2 where r is the displacement from the source point to the point where we want the field. A simple way of viewing the transition from charges to currents is to think of the current as a moving line of charge with a line charge density of lambda Coulombs/meter. In the continuous limit we can replace the sum over charges as an z integral over the charge density. This assumes that the line charge moves along the z-hat direction with a velocity v. We next show that the current is given by the product of the velocity times the linear density lambda. We do this by computing the amount of charge passing a point per unit time. This is the amount of charge that lies in a segment of length delta z which is the distance that passes the point in time t. Using this trick we have gotten rid of both the charge and the velocity and replace our sum with an integral over z times the usual current in Amperes or (Coulombs/second passing a point). For a circuit constructed from flexible wires, the z direction continually changes and we thus replace the z times z-hat vector by a path vector dL which lies along a path in the circuit. In most cases I is continuous through the circuit (remember Kirchoff's Laws). In this case the current can be taken out the integral in both the Lorentz law and the Biot-Savart law which is a considerable simplification In 3-dimensional problems it is more convenient to think of a volume charge density (rho) rather than a linear charge density lambda. In this case we replace the current by a current density vector J and integrate over a scalar volume tau rather than a path d vec-L. The current density vec-J is given by the charge density rho times the velocity vector v. We will show the current (the rate of change of charge within a volume) is given by the surface integral of J dotted in to the area vector. We use the Gauss's Law convention that d vec-a point normal and out of the volume bounded by the surface S.

Current densities



Now Δq is charge past plane in Δt.

$\Delta q = \rho\, dx\, dy\; v\; \Delta dt$

$dI = \frac{\Delta q}{\Delta dt} = \rho\, dx\, dy\; v$; $\vec{J} = \rho v\, \hat{z}$; $d\vec{a} = dx\, dy\, \hat{z}$

$\rightarrow \vec{J} \bullet d\vec{a} = \rho\, dx\, dy\; v = dI \rightarrow I = \int_S \vec{J} \bullet d\vec{a}$

We can think of $\vec{J} = \frac{dI}{da_\perp}$. Here $\vec{J} = \frac{dI}{dxdy}\hat{z}$

$\vec{J}$ units are A/m^2. Thus $\vec{F} = \int_V \vec{J} \times \vec{B}\, d\tau$

and $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} d\tau'$

Continuity Equation

$I = \int_S \vec{J} \bullet d\vec{a}$. If da⃗ points out of surface

I is current leaving volume.

Thus $I = -\frac{\partial}{\partial t}\int_V \rho\, d\tau = -\int_V \frac{\partial \rho}{\partial t} d\tau = \int_S \vec{J} \bullet d\vec{a}$

From divergence thm $\int_S \vec{J} \bullet d\vec{a} = \int_V \vec{\nabla} \bullet \vec{J}\, d\tau$

$-\int_V \frac{\partial \rho}{\partial t} d\tau = \int_V \vec{\nabla} \bullet \vec{J}\, d\tau \rightarrow \int_V \left(\vec{\nabla} \bullet J + \frac{\partial \rho}{\partial t} \right) d\tau = 0$

Since true for any volume $\boxed{\vec{\nabla} \bullet \vec{J} + \frac{\partial \rho}{\partial t} = 0}$

Surface current

$\sum_i q_i \vec{v}(\ ) \rightarrow \int_S \vec{K}\, da(\ )$

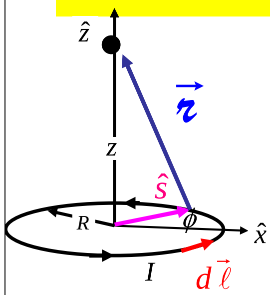
Here $\vec{K} = \sigma \vec{v} = \frac{d\vec{I}}{d\ell_\perp}$ units are $\frac{A}{m}$

$\ell_\perp$ is width of I ribbon $\perp$ to flow

7

To establish the relation between the current and J we consider J traveling along the z direction passing through an infinitesimal area vector aligned along z-hat with an area given by dx times dy. In a time dt, a charge dq will move past a plane where dq is given by the volume dx dy dz times rho. Since dz = v dt and J = rho v the current passing through the plane is J dotted into the area vector da. We can think of J as the current per transverse area a_perp. The units on J will thus be Amperes per square meter. In the current density form the expressions for the Biot-Savart and Lorentz force will involve vec-J times d tau rather than I d vec-L. The force acting on volume of moving charge is J cross B integrated over tau. Similarly the B-field created by a moving volume of charge can be written in terms of J crossed into r-hat/r^2 integrated over the volume. The conservation of current can be written in terms of J and the rate of change of charge density according to the Continuity Equation. The total integral of J dot d vec-a over a surface enclosing a volume will be the total current leaving the volume (since our da vector points outward). The total exiting current will be the negative of the integral of d q / dt in the volume where q can be written as the volume integral of the charge density. This allows us to relate the rate of change of the charge density to the divergence of J. The same continuity equation appears in other branches of physics such as fluid dynamics. The final current form we will need is the surface current or K. For example, often there is a bound current that frequently appears in magnetic materials in analogy with the bound surface charge in electric materials. Essentially K is an infinite J that only exists just below the surface. Just like I = lambda times velocity, J = rho times velocity we say K = sigma times velocity. I is the charge per unit time moving past a point, J is the charge per unit time moving past a transverse area, and K is the charge per unit time moving past a transverse line segment. K has units of Ampere/m.

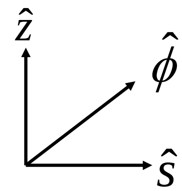
B-Field on axis of circular current loop



$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{\ell}\times\vec{\mathscr{r}}}{\mathscr{r}^3}\ ;\ \oint d\vec{\ell} = \int_0^{2\pi} R\,d\phi$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = z\hat{z} - R\hat{s}$$

$$\mathscr{r} = \sqrt{(z\hat{z} - R\hat{s})\cdot(z\hat{z} - R\hat{s})} = \sqrt{R^2 + z^2}$$



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} R\,d\phi \frac{\hat{\phi}\times(z\hat{z} - R\hat{s})}{\sqrt{R^2+z^2}^{\,3}} \xrightarrow[\hat{\phi}\times\hat{s}=-\hat{z}]{\hat{\phi}\times\hat{z}=\hat{s}} \frac{\mu_0 I}{4\pi}\int_0^{2\pi} R\,d\phi \frac{(z\hat{s} + R\hat{z})}{\sqrt{R^2+z^2}^{\,3}}$$

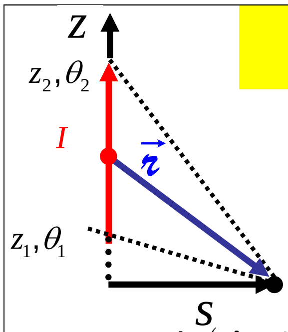
$$\int_0^{2\pi} \hat{s}\,d\phi = \int_0^{2\pi} (\cos\phi \quad \sin\phi)\,d\phi = 0$$

$$\int_0^{2\pi} \hat{z}\,d\phi = 2\pi\hat{z}$$

$$\Rightarrow \vec{B}(0,0,z) = \frac{\mu_0 I R^2 \hat{z}}{2\sqrt{R^2+z^2}^{\,3}}$$

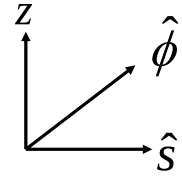
8

Although computing E-fields from Coulomb's Law might be familiar to you from previous courses, computing B from the Biot-Savart Law may not be. Perhaps the simplest example is the B-field for a point on the symmetry axis for a circular loop of radius R carrying a current I. We put the loop in the x-y plane and will calculate the field at points along the z axis. We will integrate along loop length dL by integrating the phi angle from 0 to 2 pi. The infinitesimal loop length can be written as phi-hat R d phi since the vector that points from a point at phi to a point at phi + d phi lies in the phi-hat direction. We are left with an integral involving r-hat/|r-r'|^2. Here r is "observation vector" which is the vector from the origin to the point whose field we wish to compute and r' is the "source vector" which is a vector from the origin to a point on the current loop. We write the r-vector in terms of z – the location of our observer point. We write r'-vector in terms of R and s-hat which specifies the point on the current loop and depends on phi. We evaluate the cross products using RHR to the three unit vectors described on right. We can pull out all R,s,and z factors out of the integral since they have no phi dependence. The z-hat unit vector is also constant and can be pulled out, but the s-hat integral gives zero when integrated over a complete period. We are left with the B field on for a point on the central axis. We were able to get the result very quickly since the spherical unit vectors did nearly all of the work!



Field from wire segment

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\ell} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \; ; \; \int d\vec{\ell} = \int_{z_1}^{z_2} \hat{z} dz'$$



$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = s\hat{s} - z'\hat{z}$$

$$\mathscr{r} = \sqrt{s^2 + z'^2}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} dz' \frac{\hat{z} \times (s\hat{s} - z'\hat{z})}{\sqrt{s^2 + z'^2}^{\,3}} = \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} dz' \frac{s\hat{\phi}}{\sqrt{s^2 + z'^2}^{\,3}} \xrightarrow[\text{depend on } z']{\hat{\phi} \text{ does not}} \frac{\mu_0 I s\hat{\phi}}{4\pi} \int_{z_1}^{z_2} \frac{dz'}{\sqrt{s^2 + z'^2}^{\,3}}$$

Write everything in term of θ

$$\sqrt{s^2 + z'^2} = \frac{s}{\cos\theta} \; ; \; dz' = d(s\tan\theta) = s\sec^2\theta \, d\theta = \frac{s}{\cos^2\theta} d\theta$$

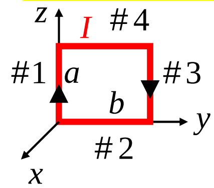
$$\vec{B} = \frac{\mu_0 I s\hat{\phi}}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{s\, d\theta}{\cos^2\theta}\right)_{dz'} \left(\frac{\cos\theta}{s}\right)^3_{\sqrt{s^2+z'^2}^{-3}} = \frac{\mu_0 I\hat{\phi}}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \; d\theta \rightarrow \boxed{\vec{B} = \frac{\mu_0 I\hat{\phi}}{4\pi s}(\sin\theta_2 - \sin\theta_1)}$$

For ∞ wire $\theta_2 = \frac{\pi}{2}$; $\theta_1 = -\frac{\pi}{2}$; $(\sin\theta_2 - \sin\theta_1) = 2 \rightarrow \vec{B} = \frac{\mu_0 I\hat{\phi}}{2\pi s}$

9

We next consider the magnetic field from a current segment. Of course we can't actually pass a current through a finite segment without violating charge conservation. But in homework you will use this straight segment result to add 4 current segments and find the field from a rectangular loop. The numerator involves a vector cross product which can be easily be simplified and is proportional to phi-hat. Since our integral is over z' and phi-hat only depends on phi, we can safely move it through the z' integrand and are left an unfamiliar integral over z'. Like many of our integrals, this one can be evaluated by a variable substitution. We switch variables from z to theta where the segment is subtended by theta_2 and theta_1. Our denominator factor is just s/cos(theta). The dz' is proportional to the derivative of a tangent and when all the dust settles we are left with the derivative of cos(theta). We are left with a simple expression in terms of the difference of the sin of our sustention angles. We can calculate the field of a polygon loop if we are willing to do the required geometry to find, theta, theta-hat, and s. If we have an infinite straight wire, our sustention angles approach +/- pi/2 and we get a simple expression for the B-field which you might recall from Physics 212 where it was done using Ampere's law. The B direction is the phi direction or the RHR direction when your thumb is along the wire. We can realize infinite wire as the limiting case of a very large square wire loop with a side "a" and a>>s.

Force, and potential of a current loop



$$\vec{F}_1 = I\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & a\\ B_x & B_y & B_z\end{vmatrix} = Ia\left(-B_y \quad +B_x \quad 0\right)$$

$$\vec{F}_3 = Ia\left(B'_y \quad -B'_x \quad 0\right)$$

$$B'_y \approx B_y + b\frac{\partial B_y}{\partial y} \ ; \ B'_x \approx B_x + b\frac{\partial B_x}{\partial y}$$

$$\rightarrow \vec{F}_1 + \vec{F}_3 = Ia\left(b\frac{\partial B_y}{\partial y} \quad -b\frac{\partial B_x}{\partial y} \quad 0\right)$$

$$\vec{F}_2 = I\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & -b & 0\\ B_x & B_y & B_z\end{vmatrix} = b\left(-B_z \quad 0 \quad B_x\right)$$

$$\vec{F}_2 + \vec{F}_4 = Ib\left(a\frac{\partial B_z}{\partial z} \quad 0 \quad -a\frac{\partial B_x}{\partial z}\right)$$

$$\rightarrow \sum_{1\rightarrow 4}\vec{F} = Iab\left(\frac{\partial B_z}{\partial z} + \frac{\partial B_y}{\partial y} \quad -\frac{\partial B_x}{\partial y} \quad -\frac{\partial B_x}{\partial z}\right)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \frac{\partial B_z}{\partial z} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_x}{\partial x}$$

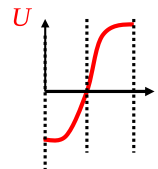
$$\rightarrow \sum_{1\rightarrow 4}\vec{F} = -Iab\left(\frac{\partial B_x}{\partial x} \quad \frac{\partial B_x}{\partial y} \quad \frac{\partial B_x}{\partial z}\right)$$

$$\vec{m} = I \times \text{area} \times \hat{n}_{RHR} = -Iab\hat{x} \rightarrow \boxed{\vec{F} = \vec{\nabla}\left(\vec{m}\bullet\vec{B}\right)}$$

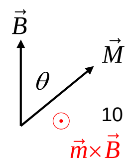
∴ No force in uniform field

If $\vec{F} = -\nabla U$ we see $\boxed{U = -\vec{m}\bullet\vec{B}}$

Increase U by increasing θ
$\rightarrow \vec{N}$ in direction of decreasing θ
or $\vec{N} \parallel \vec{M}\times\vec{B}$. We have direction, now find magnitude from work energy.



$$U = -|\vec{m}||\vec{B}|\cos\theta$$

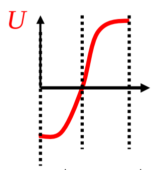


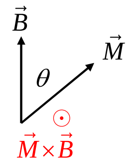
10

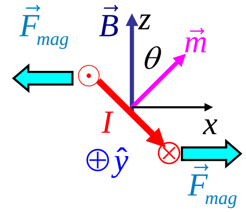
Here we find the force on an small current loop. We consider a rectangular loop in the y-z plane of dimensions b by a. Presumably a large area loop can be modeled series of these smaller loops. We expand the three fields to first order with the lowest order being defined at the origin. Our plan is to add the forces in pairs. We write L cross B for segment 1 and L cross B' for segment 2. Since there current reverses , the net force depends on the difference in the B and B' which for any component will be the (dB/dy)b. Since the length of segments 1 and 2 is "a" we have a form proportional to "a" "b" I times the dB/dy derivative. Hence we have a form proportional to the magnetic moment in the -x direction times derivatives of B for the force on the #1 + #3 segments. The force on the #2 + #4 segments is of the same form. In this case the segment lengths are "b" and the field differences are proportional to (dB/dy)a . Adding together all 4 segments we have y and z force components that depend on derivatives of Bx but an x force component that depends on derivatives of Bz and By. We can change the Fx component to a derivative of Bx using the fact that that divergence of a B field vanishes as we will establish shortly. We thus have a force given by the magnitude of the magnetic moment times the gradient of Bx. The magnetic moment is the area of the loop, points in a direction normal to the loop given by the right-hand-rule of the current flow. Since m points in the –x axis we can write the force as the gradient of m dot B. If B is constant, there will be no force on the loop. A force requires a non-uniform B field. Recall the force is negative the gradient of the potential energy and hence we have also established that the potential is – m dot B which is an extremely useful result in its own right. As an example we consider a current loop mounted on axial so it is free to rotate about the y direction. We show the direction of the currents so you can verify the direction of magnetic dipole according to the right hand rule.

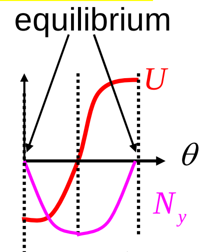
We can use the work-potential energy relation to compute the direction and magnitude of the torque acting on the wire loop. We sketch U as a function of the angle theta between B-vec and M-vec. As you can see the minimum of the potential energy is at theta=0. U increases as one increases theta. This means one must do work (eg fight the force on the coil) to increase U. The "angular" force is the torque, and evidently the torque must be in direction to decrease theta. As you can see this implies that the torque must be parallel to M-vec cross B-vec. We now have the direction and need to find the magnitude.

Finding the torque N









$|\Delta U| = \left|\frac{\partial U}{\partial\theta}d\theta\right| = \Delta\text{work} = |Nd\theta| \rightarrow |\vec{N}| = \left|\frac{\partial U}{\partial\theta}\right|$

$U = -\vec{m}\bullet\vec{B} = -|\vec{m}||\vec{B}|\cos\theta \rightarrow \left|\frac{\partial U}{\partial\theta}\right| = |\vec{N}| = |\vec{m}||\vec{B}|\sin\theta$

Hence $|\vec{N}| = |\vec{m}||\vec{B}|\sin\theta$ and $\vec{N} \parallel \vec{m}\times\vec{B} \rightarrow \boxed{\vec{N} = \vec{m}\times\vec{B}}$

$U = -\vec{m}\bullet\vec{B}\ ;\ \vec{F} = -\vec{\nabla}U$
$\vec{N} = \vec{m}\times\vec{B}$

$U = -|\vec{m}||\vec{B}|\cos\theta$

$\vec{\mathrm{N}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ m\sin\theta & 0 & m\cos\theta \\ 0 & 0 & B \end{vmatrix} = -mB\sin\theta\,\hat{y}$

Interestingly enough, in the large distance limit $\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\bullet\hat{r})\hat{r} - \vec{m}}{r^3}$

which is exactly the same as the form for electric fields $\frac{\mu_0}{4\pi} \rightarrow \frac{1}{4\pi\varepsilon_0}; \vec{m} \rightarrow \vec{p}\ ;\ B \rightarrow E$

For circle loop in x-y plane $\vec{m} = I\pi R^2\hat{z}$ if $\vec{r} = Z\hat{z}$ and r $\gg$ R

$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I\pi R^2\hat{z}}{Z^3} = \frac{\mu_0}{2}\frac{IR^2\hat{z}}{Z^3}$ same as BS result

11

To find the magnitude of the torque, we set the work done rotating the coil against the torque to the change in potential energy. Work is force time distance or in angular problems: torque times angle. We thus get that the magnitude of the torque is the derivative of U with respect to theta. Hence the magnitude of the torque is the product of the field magnitude times the magnetic moment magnitude times the sine of the angle between them. Recall this is the standard expression for the magnitude of a cross product. The direction of the torque is along m-vec cross B-vec. If N-vec = m-vec cross B-vec we get the direction and magnitude correct and hence this must be the correct torque expression. We plot the torque and potential energy as a function of function of theta. We also plot the torque as a function of theta which is along the axle or y-direction. We see that when the magnetic moment is aligned with the B-field at theta =0, the coil is in a stable equilibrium position with zero torque. If we move the coil at some positive theta, a negative y torque will be present which rotate the coil back to the bottom of the potential hill at theta = 0. You can see the way the magnetic forces on the loop segments directed to the left and right create this torque in the direction of the equilibrium. With no friction the coil will oscillate back and forth about the stable equilibrium with an approximately constant frequency. The maximum torque magnitude occurs where the potential hill is the steepest at theta = 90 degrees. This is also borne out by the torque equation which shows the torque is proportional to sin(theta). There is also another equilibrium point where no torque exists at theta = 180 degrees corresponding to the case where the magnetic moment is anti-parallel to the external B-field. Any slight nudge will cause the coil to try to flip over and oscillate about theta =0. Finally we point out an interesting parallel between the electric dipole moment and the magnetic dipole moment. The expressions for the field (a long distance away) are basically the same with the indicated “translations”. As a quick check we consider the magnetic moment for a circular loop in the x-y plane. We use B-vec expression to compute the field on a point on the z-axis. Indeed we recover the Biot-Savart expression in the limit Z>>R.

Differential magnetostatics

Recall $\varepsilon_0 \vec{\nabla}\bullet\vec{E} = \rho$ what is $\vec{\nabla}\bullet\vec{B}$?

$$\vec{\nabla}\bullet\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\bullet\left[\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right]d\tau'$$

We use the product rule:

$$\vec{\nabla}\bullet\left[\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right] =$$

$$\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\bullet\vec{\nabla}\times\vec{J}(\vec{r}\,') - \vec{J}(\vec{r}\,')\bullet\left[\nabla\times\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right]$$

$$\vec{\nabla}\times\vec{J}(\vec{r}\,') = \left(\frac{\partial}{\partial x}\quad\frac{\partial}{\partial y}\quad\frac{\partial}{\partial y}\right)\times\vec{J}(\vec{r}\,') = 0$$

Since $\vec{J}(\vec{r}\,')$ only depends on

$(x' \quad y' \quad z')$ and $\frac{\partial x'}{\partial x} = 0$ etc

$\vec{\nabla}\times\vec{E}_{Coul} = 0 \rightarrow \vec{\nabla}\times\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = 0$

$\rightarrow \vec{\nabla}\bullet\left[\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right] = 0$ and $\vec{\nabla}\bullet\vec{B} = 0$

in magnetostatics. Actually $\boxed{\vec{\nabla}\bullet\vec{B} = 0 \text{ always}}$

What is $\vec{\nabla}\times\vec{B}$?

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left[\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right]d\tau'$$

We use the product rule:

$$\vec{\nabla}\times\left[\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right] = \vec{J}(\vec{r}\,')\vec{\nabla}\bullet\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) - \left(\vec{J}(\vec{r}\,')\bullet\vec{\nabla}\right)\left[\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right]$$

12

As was the case with electrostatics, magnetostatics can be put in differential form. For example one can compute the divergence of B from the Biot-Savart law. We need to take the divergence of the J term as well as the (r-r')/|r-r'|^3 term. But J(r') only involves source positions and the del operator is with respect to "observer" variables, so del sails through J(r') and operates just on (r-r')/|r-r'|^3 through the cross product. But the curl of (r-r')/|r-r'|^3 is zero since this is the form used in Coulomb's law and we know that the curl of Coulomb law fields is zero. This means the divergence of B-fields is zero. Since divergence of E is the electric charge density, this fact strongly suggests that there is no magnetic charge density and, in fact, all attempts to search for magnetic charges (also called magnetic monopoles) have failed. In fact even with time varying currents and fields, the B-field has zero divergence. We can use the same technique to find the curl of B. We now have the curl of a curl and there is a specific product rule (Griffiths Chp 1) we can use. Again del sails through J and only operates on (r-r')/|r-r'|^3 .

Ampere's Law

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\vec{\nabla}\cdot\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)d\tau' - \frac{\mu_0}{4\pi}\int\left(\vec{J}(\vec{r}\,')\cdot\vec{\nabla}\right)\left[\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right]d\tau'$$

From Gauss's Law lect $\vec{\nabla}\cdot\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}=4\pi\delta^3\left(\vec{r}-\vec{r}\,'\right)$

Hence 1st term becomes

$$\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\vec{\nabla}\cdot\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau' = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')4\pi\delta^3\left(\vec{r}-\vec{r}\,'\right)d\tau' = \mu_0\vec{J}(\vec{r})$$

Interestingly enough the 2nd term vanishes.

Thus $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$ or integrating over area

$$\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{\ell} = \int_S \mu_0\vec{J}\cdot d\vec{a} = \mu_0 I_{\text{enc}}$$

$\rightarrow \boxed{\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{\text{enc}}}$ familiar Ampere's Law

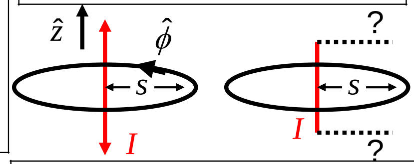
Ampere's law is often easiest way to find $\vec{B}$ in symmetric problems

Ex: $\vec{B}$ for infinite wire

If we assume $\vec{B} = B(s)\hat{\phi}$

$\oint \vec{B}\cdot d\vec{\ell} = 2\pi s B = \mu_0 I_{\text{enc}} = \mu_0 I$

or $B = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ agrees w/ Biot-S



But clearly $B = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ can't be right for a short current segment where

Biot gives $\vec{B}=\frac{\mu_0 I\hat{\phi}}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)$

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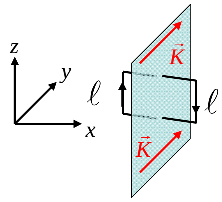
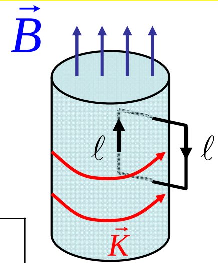
Whats wrong?

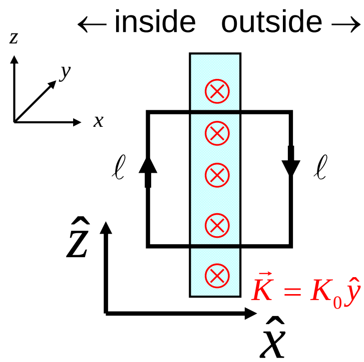
We have seen the divergence of (r-r')/|r-r'|^3 before in our discussion of Gauss's Law in differential form. We found that it was a 3-dimensional delta function. This forces the integral over J at the source point to return J at the observation point as well as conveniently canceling a 4 pi in the denominator of the first term. We thus have the elegant result that the curl of B is mu_0 J. Fortunately the second term vanishes and the elegance survives.

We can next apply Stokes' law to this result by taking the line integral of each side. On the left we have the line integral of the B-fields, on the right we have the surface integral of the current density J which , as we saw earlier, is just the current enclosed in the loop. This is the (hopefully) familiar Ampere's Law which you saw in Physics 212. Ampere's law is the tool of first choice for computing a B field in the highly symmetric situations where a path can be found that is proportional to the full B-field magnitude. A great example is the B-field of an infinite wire at a point a distance s from the wire center. If we assume that the B-field is in the phi-direction and can only depend on the radial distance s, we easily get the form of B(s) since the line integral is just 2 pi s B_phi which must equal the mu_0 I. Of course one should worry about our assumptions on the form of the B-field when applying Ampere's Law to find the field. We get some insight from the symmetry of the problem and the vanishing of the divergence. We also know that all three components of the curl must vanish in field free regions. From azimuthal symmetry of the source it is clear that B cannot depend on phi. The Biot-Savart Law tells us that B points in the z-hat cross (r-r') direction and thus must be perpendicular to the z-axis and thus has no Bz. It is equally clear that it cannot depend on z since an infinite wire is looks exactly the same at any finite z. The vanishing of the z component of the curl means we must have d(s*B_phi)/ds = 0. This allows for a B_phi = constant/s and/or a B_s= constant'/s. The single wire is of the form B_phi = constant/s. We will show shortly (and you probably recall) that a solenoid will have a constant B_z. The finite wire shows that one can easily get mislead by an Ampere's law argument. For example a small, centered current segment looks like it will just have a non-zero B_phi. It seems like the same Ampere's law treatment should work for the finite current segment as the infinite current segment and we should get the same answer. Of course out Biot-Savart calculation shows us this is wrong since a smaller segment will have a smaller sin(theta_2)-sin(theta_1) and hence the field of this small current segment must be smaller than the infinite current segment. It might be nice to test this experimentally but it seems impossible to actually build a magnetostatic finite current segment!

Ampere Law prototypes







Current sheet

$\vec{K} = K_0\hat{y} \qquad (K \text{ in } A/m)$

$\oint \vec{B}\bullet d\vec{\ell} = 2B\ell = \mu_0 I_{enc} = \mu_0 \ell K \; ; \; B = \dfrac{\mu_0 K}{2}$

$\vec{B}(x>0) = -\dfrac{\mu_0 K_0 \hat{z}}{2} ; \vec{B}(x<0) = +\dfrac{\mu_0 K_0 \hat{z}}{2}$

Using normal $\hat{\eta} = \hat{x}$ we have useful a BC form $\vec{B}_> - \vec{B}_< = \mu_0 \vec{K} \times \hat{\eta}$ where $\vec{B}_> = \vec{B}(x>0)$; $\vec{B}_< = \vec{B}(x<0)$

Solenoid

$\vec{K} = K_0\hat{\phi}$

$\vec{B}$ only exists inside solenoid

$\oint \vec{B}\bullet d\vec{\ell} = B\ell = \mu_0 I_{enc} = \mu_0 K_0 \ell$

$\rightarrow \vec{B} = \mu_0 K_0 \hat{z}$

Often $K = nI$ where $n =$ turn/m

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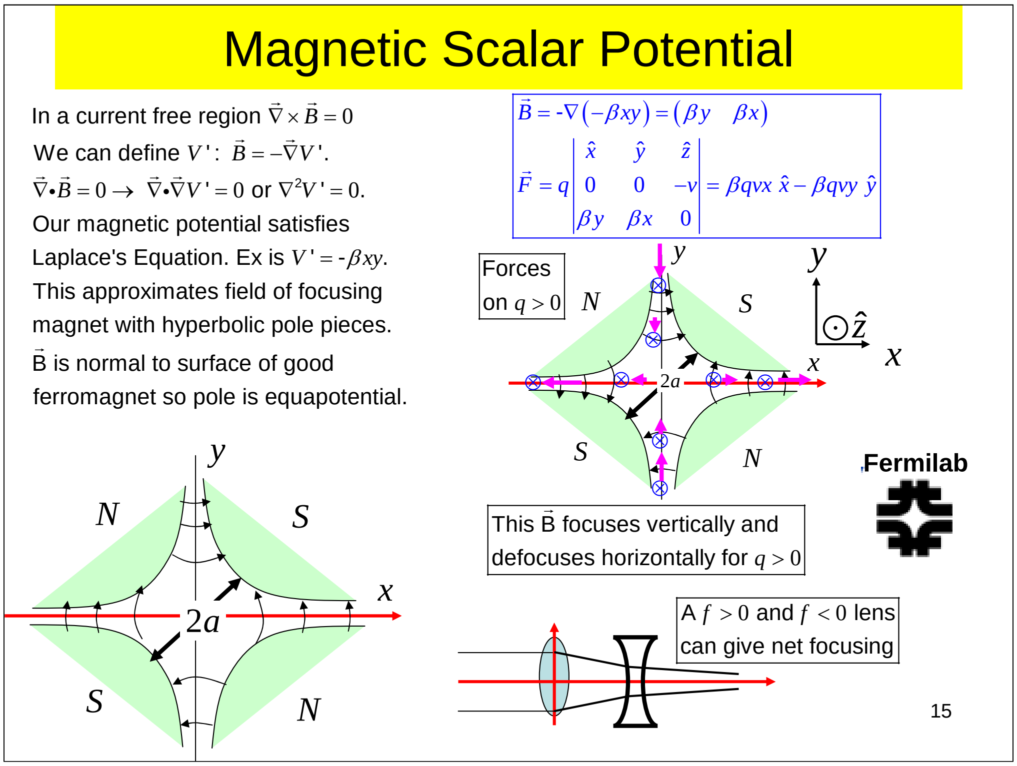
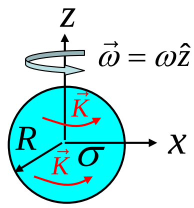
There are many other examples where we get a very simple Ampere's Law solution besides the infinite wire. You are familiar with many of these from Physics 212. The two that are illustrated here are the current sheet where there is a constant surface current along an infinite plane. Here we use an Ampere's Loop that is transverse to the plane. The only unfamiliar thing might be the realization that the total current that cuts the loop is K times L_perp where L_perp is in the surface and transverse to K. This is easily remembered one you realize that K is measured in amps/meter. The B-field is transverse to K and the plane normal and switches direction as you pass through the plane. The current sheet is the magnetic analogy to the infinite plane of charge in electrostatics. The other common example is the infinite solenoid. Here we write the current as a surface current K phi-hat although one typically gets the surface current by winding current carrying wires around a cylinder. To progress with Ampere's law we need to realize that there is no B-field outside the solenoidal cylinder. Here the L_perp used to compute the total current through the loop is directed along the z or symmetry axis. We can also use the current sheet Ampere Law construction to establish an important boundary condition which relates the B field on either side of the current sheet to the cross product of the surface current K and the normal to the surface eta-hat. In our example eta-hat is x-hat and K-vec points along y-hat. Our BC condition is on B_> which is the field past the plane in the eta-hat direction, and B_< which is the field past the plane in the negative eta_hat direction. The B_> and B_< difference is given by mu_0 times the cross product of K and eta. Indeed this is true for the current sheet alone and will be true for any B-field on either side of a current sheet. We will use this frequently. You can think of a solenoid as a rotated version of two current sheets on either side of the diameter with oppositely directed K's. You can convince yourself that two opposite current and displaced current sheets only have a field between them.



Just like it is possible to define a potential for electric fields such that E = - gradient V, it should be possible to define a magnetic potential such that B = - grad V' at least in regions where the curl of B vanishes (eg regions free of currents). Since the divergence of B always vanishes, if B = - grad V' , V' will satisfy the Laplace Eq. In the Laplace Chp. we showed V=-beta x y satisfies Laplace's Eq since the Laplace Eq. involves double derivatives of x, and y and this potential form only involves single derivatives and thus vanishes. In the next chapter we will discuss the boundary conditions for magnetic fields in matter but the magnetic fields at the surface of highly magnetic ferromagnetic materials are very nearly perpendicular to the material surface. This means the iron material in the pole pieces of a magnet look very much like the equapotentials that conductors form in electrostatic. As you can see, the full magnet has 4 magnetic poles and are therefore known as quadrupole magnets. In the case V'=-beta x y , the constant magnetic potential lines look like hyperbolas. If we include the coils used to energize the magnets we get the stylistic profile that appears in the Fermilab Logo. Why are these "hyperbolic" magnets so important in accelerators? We get a clue by computing the form of the magnetic field by taking the gradient of the vector potential and use the B-field to compute the forces acting on a charge particle traveling along the x-axis. We find that the x force is proportional to x and the y force is proportional to y with the opposite sign. Here x and y are with respect to the center of the magnet. The reason the field grows stronger as the particle gets further from the surface is because the pole pieces get closer together and the magnetic field lines get denser. A system that deflects particles in a way proportional to their distance acts like a lens. If you do the geometry you will see a parallel beam of particles gets deflected into a single focal point. One complication is that the quadrupole magnet focuses in the x – direction and defocuses in the y-direction (or vice versa depending on the charge). However a converging of focal length f followed by a diverging lens of focal length –f can act as a net converging lens if they are placed apart at the right distance compared to the focal length f. Why is it important to have magnetic focusing in an accelerator. For example, at Fermilab protons from a hydrogen bottle, travel for 10 or so seconds just beneath the speed of light through the mile circumference accelerator ring. The would clearly wonder out and strike the beam pipe without the magnetic focusing provided by the alternating quadrupole magnets. It wasn't until the early 1950's that the strong focusing principle made possible with quadrupole magnets was incorporated into accelerators such as Fermilab. Although the magnetic potential V' is a very useful concept which is in strong analogy with the electrostatic potential, it is important to realize that it has no connection to a charge particle energy. After all static magnetic fields do no work and hence the change in magnetic potential is not the work done on particles. There is another magnetic potential called the *vector potential* which does have "mechanical" significance as we will show and appears in Lagrangean

Spinning Ball of Charge w/ Magnetic Scalar Potential



$\vec{K} = \sigma\vec{v} = \sigma\,\vec{\omega}\times\vec{r}$ where $\vec{\omega} = \omega\hat{z} \rightarrow$

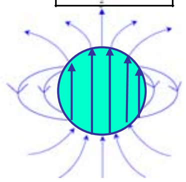
$\vec{K} = \sigma\omega R\;\hat{z}\times\hat{r} = \sigma\omega R(\cos\theta\hat{r} - \sin\theta\,\hat{\theta})\times\hat{r} = \sigma\omega R\sin\theta\,\hat{\phi}$

Try $V' = \left(Ar + \frac{B}{r^2}\right)P_1(\cos\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$. To keep V' finite

$V'(r<R) = Ar\cos\theta$ & $V'(r>R) = \left(B/r^2\right)\cos\theta$.

To find A & B we use $\vec{B}_> - \vec{B}_< = \mu_0\vec{K}\times\hat{\eta} = \mu_0\vec{K}\times\hat{r}$

$\nabla\bullet\vec{B} = 0$



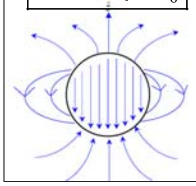
$$\boxed{\vec{B}_> - \vec{B}_< = \mu_0\sigma\omega R\sin\theta\;\hat{\phi}\times\hat{r} = \mu_0\sigma\omega R\sin\theta\,\hat{\theta}}$$

$$\vec{B} = -\nabla V' = -\hat{r}\frac{\partial V'}{\partial r} - \frac{\hat{\theta}}{r}\frac{\partial V'}{\partial\theta} \rightarrow \boxed{\vec{B}_< = -A\cos\theta\;\hat{r} + A\sin\theta\;\hat{\theta}}$$

$$\boxed{\vec{B}_> = \frac{2B}{R^3}\cos\theta\;\hat{r} + \frac{B}{R^3}\sin\theta\;\hat{\theta}}$$

$$\vec{B}_> - \vec{B}_< = \left(\frac{2B}{R^3} + A\right)\cos\theta\;\hat{r} + \left(\frac{B}{R^3} - A\right)\sin\theta\;\hat{\theta} = \mu_0\sigma\omega R\sin\theta\,\hat{\theta}$$

$\nabla\bullet\vec{E} = \rho/\varepsilon_0$



$$\rightarrow \frac{2B}{R^3} + A = 0 \rightarrow B = \frac{-AR^3}{2} \text{ and } -\frac{3A}{2} = \mu_0\sigma\omega R \rightarrow A = -\frac{2}{3}\mu_0\sigma\omega R$$

$$\rightarrow \boxed{\vec{B}_< = \frac{2}{3}\mu_0\sigma\omega R\left(\cos\theta\;\hat{r} - \sin\theta\;\hat{\theta}\right) = \frac{2}{3}\mu_0\sigma\omega R\;\hat{z}\text{ constant!}}$$

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Here is a more interesting scalar potential example where we recycle our spherical separation of variable solution in a way very close to the “glued” charge example. In this case we spin the ball of with surface charge density sigma around the z-axis with an angular velocity omega. This will create a surface current given by sigma times velocity where you might recall from Physics 425 is given by omega cross r-vec where r-vec is from the origin to a point on the sphere of radius R. Multiplying this out we find that the surface current is in the phi-hat direction and is proportional to sin(theta). As a guess we try the Laplace potential for P1 (cos theta) since this will result in a simple function of cos(theta). Just like the P1 glued charge example, we will have an A term which is proportional to r for the inside and a term proportional to B/r^2 on the outside. The V’> and V’< are limited to either A or B since we don’t want V’ to blow up at r=0 or r= infinity. We need two BC to fix the unknown A and B variables. Interestingly enough, the potential V’ is not continuous at r=R unlike the case of the spherical electret. Rather, we get the two BC from our “current sheet” expression which relates B just below B< and just above B> the current sheet in the r-hat direction. The K-vec cross eta-hat lies in the theta direction which means that the radial component of B is continuous and the theta component is not. We apply this by first evaluating the B field r-hat and theta-hat components by evaluating the spherical gradient expression at the current sheet location of r=R. Since the gradient gives B-theta component proportional to sin(theta) for both B< and B> and the K cross eta is also proportional to sin(theta) we know we will have a valid solution, the sin(theta) factors will cancel, and thus our V’ propto P1 guess was correct! Essentially continuity of the radial components of B allows us to write B in terms of A and R and the discontinuity of the theta components allows us to find A in terms of sigma, R, and omega. We can insert this A value into our B< expression to get the B-field inside the sphere, and we find that (1) the B-field is in the z-hat direction which is the direction of angular velocity omega and (2) is uniform inside the sphere. It is not that surprising that B is along the z-axis for points on the z-axis since we can think of the sphere as a collection of circular loops -- each centered on the z-axis. By the right-hand rule it is clear that a positive current implies a positive B_z in the center of each circular loop. It is surprising that the field is **uniform** within the sphere, just like a solenoidal B-field. Unlike the solenoid, the rotating charge field leaks both inside and outside the sphere and hence we cannot use a simple Ampere’s law argument to decide that B is uniform. I also include a sketch of the magnetic field lines for the spinning charge and compare it to the spherical electret example from the last chapter. The big difference is that B-field lines inside the sphere for the magnetic case are in the **same direction** as those just outside the north or south pole, while the E-field lines inside the spherical electret are in the **opposite direction** as those just outside either pole. It is easy to see why. In the electret case, there are electric charges on the surface which means field lines can end and begin on a charge and by Gauss’s law one can show E_r is discontinuous. In the magnetic case, there are no “magnetic” charges which can start or stop a magnetic field line, which by a Gauss’s law argument applied to Del dot B = 0 implies B_r is continuous. Incidentally there is **never** a magnetic charge in any physical system to the best of our knowledge!

Analogy w/ electric dipole potential

$$V(\vec r > R) = \frac{\hat r \bullet \vec p}{4\pi\varepsilon_0 r^2} \xrightarrow[1/\varepsilon_0 \to \mu_0]{\vec p \to \vec m} \text{Does } V'(\vec r) = \frac{\mu_0 \hat r \bullet \vec m}{4\pi r^2} \ ?$$

$V'(r < R) = Ar\cos\theta$ & $V'(r > R) = \frac{B}{r^2}\cos\theta$

$A = -\frac{2}{3}\mu_0\sigma\omega R$; $B = \frac{-AR^3}{2} = \frac{\mu_0\sigma\omega R^4}{3}$

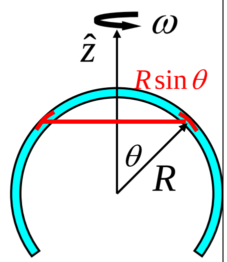
$V'(r > R) = \frac{\mu_0\sigma\omega R^4\cos\theta}{3r^2} \xrightarrow{\vec m \| z} \frac{\mu_0 |\vec m| \cos\theta}{4\pi r^2}$

Is $\boxed{\vec m = \frac{4\pi\sigma\omega R^4 \hat z}{3}}$ Check with $d\vec m = \hat z\ (Area)dI$

$dI = \vec K\ d\left|\vec\ell_\perp\right|$

$d\vec\ell_\perp = \hat\theta\ Rd\theta$

$\vec K = \sigma\omega R\sin\theta$



$dI = d\ell_\perp K = Rd\theta\left(\sigma\omega R\sin\theta\right)$; $dm = \hat z\left[\pi(R\sin\theta)^2\right]_{\text{Area}} dI = \hat z\pi\sigma\omega R^4\sin^3\theta\ d\theta$

$\vec m = \hat z\pi\sigma\omega R^4\int_0^\pi (1-\cos^2\theta)\sin\theta\ d\theta \xrightarrow{u=\cos\theta} \vec m = \hat z\pi\sigma\omega R^4\int_{-1}^{1}(1-u^2)du$

$\vec m = \hat z\pi\sigma\omega R^4\left[u - \frac{u^3}{3}\right]_{-1}^{1} \to \vec m = \frac{4\pi\sigma\omega R^4}{3}\hat z$ which checks!

17

Recall we obtained the a elegant expression for the potential for a glued ball of charge with sigma proportional to cos(theta) in our Laplace chapter. We will show a spinning ball of charge has a magnetic moment. Wouldn't it be cool if we could make the obvious electric to magnetic transformation to our charged ball expression to get the magnetic scalar potential for a spinning ball? Lets assume that this actually does work and use our spinning ball magnetic potential to compute the required magnetic moment. We get an boxed expression for vec-m. We can check this by thinking of the spinning ball being equivalent to elements of current loops each with a magnetic moment element along the z-hat direction. The infinitesimal magnetic moment will be d vec-m = d i (area) z-hat = d i [pi (R sin theta)^2] z-hat since R sin theta is the equivalent radius of the loop. A current element d i is given by the magnitude surface current K times d Lperp. For this case d Lperp is in the theta-hat direction and has a length R d theta. We know the surface current for the spinning charged ball from the previous slide and hence we have a current element which goes as sin^2(theta) d (theta) and a magnetic moment element which is proportional to sin^3 (theta) d(theta). We can evaluate the integral of our magnetic moment elements from 0 to pi using the usual u = cos(theta) variable substitution for spheres. Indeed we get the same vec-m that we need so that the scalar potential expression in terms of vec-m is analogous to the voltage expression for for an vec-p electric dipole moment. The electric-magnetic symmetry is powerful!

Magnetic Vector Potential

In addition to $\vec{B} = -\vec{\nabla} V'$ potential we have vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$.

In static problems $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{J} \rightarrow \vec{\nabla} \bullet \left(\vec{\nabla} \bullet \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Interestingly we can alway find $\vec{A}$ such that $\vec{\nabla} \bullet \vec{A} = 0$ using Gauge Freedom .

This exploits $\vec{\nabla} \times \left(\vec{\nabla} \phi\right) = 0$ for any $\phi(\vec{r})$

If $\vec{\nabla} \bullet \vec{A} = f(\vec{r})$ we find new $\vec{A}' = \vec{A} + \vec{\nabla} \phi$

$$\vec{\nabla} \bullet \vec{A}' = \vec{\nabla} \bullet \left(\vec{A} + \vec{\nabla} \phi\right) = f(\vec{r}) + \nabla^2 \phi = 0$$

as long as $\nabla^2 \phi = -f(\vec{r})$. Equiv to Poisson Eq (eg finding potential for a charge distrib) and thus a $\phi(\vec{r})$ solution always exists

We thus have $-\nabla^2 \vec{A} = \mu_0 \vec{J}(\vec{r})$

In Cartesian Coords this is three Poisson Eq. $\nabla^2 \vec{A}_x = -\mu_0 \vec{J}_x$

$\nabla^2 \vec{A}_y = -\mu_0 \vec{J}_y$; $\nabla^2 \vec{A}_z = -\mu_0 \vec{J}_z$

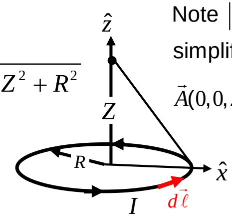
We can thus recycle superposition $V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{\mathscr{r}} d\tau'$ or solution to $\varepsilon_0 \nabla^2 V = \rho(r')$ to get $\boxed{\vec{A}(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')}{\mathscr{r}} d\tau'}$ as solution to $\nabla^2 \vec{A} = -\mu_0 \vec{J}$. Two caveats:

(1) No $\vec{J}$ at $\vec{r}' \rightarrow \infty$

(2) Only true in Cartesian coords.

An example is to find $\vec{B}(0,0,Z)$ from $\vec{A}$: $\vec{A}(r) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\ell}}{\mathscr{r}}$ for circular loop.

$\mathscr{r} = \sqrt{Z^2 + R^2}$



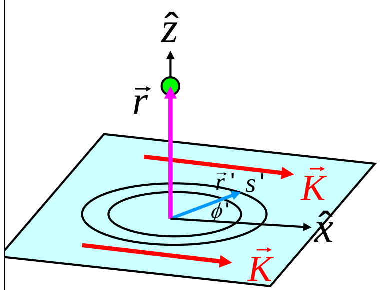
Note $|\vec{r} - \vec{r}'|$ is constant so simplifies

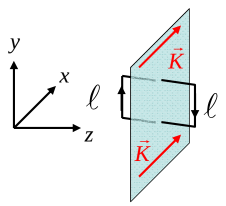
$$\vec{A}(0,0,Z) = \frac{\mu_0 I}{4\pi\sqrt{Z^2 + R^2}} \oint d\vec{\ell} = 0$$

18

Although the scalar potential can be very useful in practical problems, a more useful theoretical concept is called the vector potential. Here the magnetic field is not the gradient of a scalar potential but the curl of a vector potential A. As we will see shortly, the vector potential is the way magnetic forces are incorporated in advanced mechanics (Lagrangian mechanics), quantum mechanics and the Standard Model. We get the equation for the vector potential in magnetostatics directly from the differential form of Ampere's law. The curl of the curl can be written in the indicated form using one of the Griffiths Chp. 1 identities. Interestingly enough, it is always possible to eliminate the first term through a gauge transformation. As you recall traditional scalar potentials are not uniquely determined – you can always add a constant to potential and generate the same fields or equations of motion. The freedom in defining the vector potential is rather staggering. You can add the gradient of any function to a vector potential and get a new vector potential that corresponds to the same magnetic field but might look very different from the original. This is because the curl of a gradient is always zero. The slide shows us how to find the gauge transformation that will eliminate the first term in the vector potential equation. It shows that if one adds the gradient of a function phi(r) to the original vector potential to form a new A' and if one chooses the Laplacean of phi to equal the negative of the divergence of A, then del dot A will equal zero. Of course in principle it might not be possible to solve the equation for the hypothetical gauge function phi. But this equation is exactly the same as the Poisson Eq. used to find the electrostatic potential for an arbitrary charge density. So we know a solution must always exist. Interestingly enough, once we eliminate the first term by working in the Coulomb gauge (which is the name of gauge where the divergence of A vanishes) we are left with three Poisson Eq. – one for each Cartesian component of the vector potential. This allows us to recycle our superposition expression for the 1/(r-r') potential for an element of charge used in electrostatics. We need to stick to Cartesian coordinates since we want to factor the unit vectors out of the integral and hence they cannot depend on position as would be the case in curvilinear coordinates. We also need all currents to disappear at infinity so that we can use infinity as our potential reference point. Although we are able to get a very elegant looking integral expression each A component (that looks nearly identical to the scalar potential of electrodynamics), the bad news is that it isn't as easy to deal with as the Biot-Savart law if we are interested in computing the field from a current distribution which isn't symmetric enough to use Ampere's law. We illustrate the difficulty by considering the vectpr potential of a circular current loop for a point on the axis of the loop. The problem starts off simply enough. Our integral consists of 1/|r-r'| which is independent of our variable of integration which is dL= (Rcos(phi),Rsin(phi),0) dphi and can therefore be factored out of the integral. But the integral of the path dL around a complete path is zero! Hence we very quickly are able to find A(0,0,Z) but finding A on the z axis does us little good in our quest to compute B on the axis. We will show how to get A for a loop by expanding | r – r'| later.

The vector potential of a current plane





$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\int \frac{\vec J(r')}{\mathscr{r}}\, d\tau' \to \vec A(\vec r) = \frac{\mu_0}{4\pi}\int \frac{K_0 \hat x}{\mathscr{r}}\, da'$$

$$\mathscr{r} = |\vec r - \vec r'| = \sqrt{(z\hat z - s'\hat s)\bullet(z\hat z - s'\hat s)} = \sqrt{s'^2 + z^2}$$

$$da' = 2\pi s' ds' \to \vec A(r) = \frac{\mu_0 K_0 \hat x}{4\pi}\int_0^\infty \frac{2\pi s' ds'}{\sqrt{s'^2+z^2}} \quad \text{Let } v = s'^2 + z^2$$

$$dv = 2s' ds \to \int_0^\infty \frac{s' ds'}{\sqrt{s'^2+z^2}} = \frac{1}{2}\int_{z^2}^\infty v^{-1/2} dv = \left[v^{+1/2}\right]_{z^2}^\infty = -|z| + \infty$$

$$\to \vec A(z) = -\frac{\mu_0 K_0 \left(|z| - \infty\right)}{2}\hat x$$

$$\to \vec B(z>0) = -\frac{\mu_0 K_0}{2}\begin{vmatrix} \hat x & \hat y & \hat z \\ \partial_x & \partial_y & \partial_z \\ z-\infty & 0 & 0 \end{vmatrix} = -\frac{\mu_0 K_0 \hat y}{2}$$

$$\to \vec B(z<0) = -\frac{\mu_0 K_0}{2}\begin{vmatrix} \hat x & \hat y & \hat z \\ \partial_x & \partial_y & \partial_z \\ -z-\infty & 0 & 0 \end{vmatrix} = +\frac{\mu_0 K_0 \hat y}{2}$$

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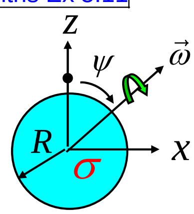
In this example we use the vector potential integral to compute the magnetic field of a uniform current sheet. We place our observation point along the z-axis and our source point at s' s-hat. Our vector potential integral becomes an area integral of the form 2 pi s' ds'. This integral is easy to evaluate by a variable substitution. For a z>0 point our integral is of the form z – infinity where the infinity part comes from the infinite s' extent. Having a divergent A-vec is a bit disconcerting but doesn't cause any problem in the B-field calculation since the B-field is the curl of A which involves derivatives with respect to observer coordinates and our "infinity" does not depend on these coordinates. Instead we get the expected B-field on either side of an infinite current sheet. We will use a similar technique to calculate electromagnetic radiation from an oscillating current sheet in Physics 436.

Vector potential of spinning ball of charge

Griffiths Ex 5.11



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}da'$$

Griffiths ex 5.11 does this integral for rotating sphere

and gets: $\vec{A}(|\vec{r}| < R) = \frac{\mu_0 R\sigma}{3}\vec{\omega}\times\vec{r} \;\rightarrow \vec{B}(|\vec{r}| < R) = \frac{2\mu_0 R\sigma}{3}\vec{\omega}$

$\vec{A}(|\vec{r}| > R) = \frac{\mu_0 R^4\sigma}{3r^3}\vec{\omega}\times\vec{r}$ using $\vec{m} = \frac{4\pi\sigma\vec{\omega}R^4}{3}$

we get $\vec{A}_> = \frac{\mu_0\vec{m}\times\hat{r}}{4\pi r^2} \Leftrightarrow V_>' = \frac{\mu_0\vec{m}\bullet\hat{r}}{4\pi r^2}$ Analogy w/ V' is striking!

Both $\vec{A} = \frac{\mu_0\vec{m}\times\hat{r}}{4\pi r^2}$ and $V' = \frac{\mu_0\vec{m}\bullet\hat{r}}{4\pi r^2}$ are exact if we are outside the spinning ball. We will show that arbitrary current loop can be multipole expanded as $\boxed{\vec{A}(r >> R) = \frac{\mu_0\vec{m}\times\hat{r}}{4\pi r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)}$.

20

In Griffiths Ex 5.11, the text uses the vector potential integral to evaluate vec-A for the spinning ball of charge and takes its curl to find B for a point inside and outside of the ball. Of course, the text gets the same answer we did for the B-field as we did using our Laplace Eq. treatment using the scalar potential. I find the Laplace Eq. treatment to be easier since the vector potential evaluation that Griffiths uses rather tricky – he is the king of spherical integrals. We can use our magnetic moment expression to convert from the angular velocity to the magnetic moment. When written this way we see a remarkable analogy between Griffiths expression for the vector potential and our scalar potential V' expression. For the scalar potential we have the dot product between m-vec and r-hat , while for the vector potential we have the cross product between m-vec and r-hat. In both the glued charge, and spinning ball examples are **ideal** electric and magnetic dipoles and hence our simple vector and scalar potential expressions are **exact**. In general we can expand the vector potential for an arbitrary current system in a magnetic multipole expansion which we develop at the end of these lectures. The magnetic moment or (magnetic dipole) term is the leading term in this expansion since presumably there are no magnetic monopoles.

The vector potential of a long solenoid

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}\,da'$ is a difficult integral .

But for finite length solenoid, we expect $\vec{A}(\vec{r})$ to be in direction of $\vec{K}(\vec{r}\,')$ or $\vec{A} \parallel \hat{\phi}$.

Symmetry $\vec{A}$ cannot depend on ϕ or $z \to \vec{A}$ (s) $\propto \hat{\phi}$.

We use a Stokes' law trick $\vec{\nabla}\times\vec{A} = \vec{B}$

$\oint \vec{A}\bullet d\vec{\ell} = A_\phi 2\pi s = \int_S \vec{\nabla}\times\vec{A}\bullet d\vec{a} = \int_S \vec{B}\bullet d\vec{a} \equiv \Phi_m$

Recall $\vec{B} = \mu_0 K_0 \vec{z}$ where $\vec{K} = K_0\hat{\phi}$ if inside.

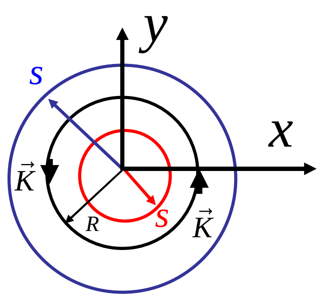
For $r < R$; $\Phi_m = B\pi s^2$; $A_\phi 2\pi s = B\pi s^2 \to \vec{A} = \frac{\hat{\phi}\,Bs}{2}$

For $r > R$; $\Phi_m = B\pi R^2$; $A_\phi 2\pi s = B\pi R^2 \to \vec{A} = \frac{\hat{\phi}\,BR^2}{2s}$

$\vec{\nabla}\times\vec{A} = \frac{\hat{z}}{s}\frac{\partial}{\partial s}\left(sA_\phi\right) = \frac{\hat{z}}{s}\frac{\partial}{\partial s}\left(s\frac{Bs}{2}\right) = B\hat{z}$ for $(s < R)$

$\vec{\nabla}\times\vec{A} = \frac{\hat{z}}{s}\frac{\partial}{\partial s}\left(s\frac{BR^2}{2s}\right) = 0$ for $(s > R)$

Interesting that $\vec{A}(s > R) \neq 0$ although $\vec{B} = 0$



But why bother calculating A if we already know B?

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The vector potential for an infinite solenoid is much easier! Its difficult to use our “Coulomb” potential integral. But we note that all of the currents are in the phi direction and our integral expression suggests that A should be in the direction of the current. Perhaps A is in the phi-hat direction. The magnitude of A cannot depend on either phi or z which suggests (1) A only has a phi component and (2) A only depends on s. If so we should be able to get A very quickly by applying the Stokes’ Theorem to the curl of A equals B. This means the line integral of A (or 2 pi s A_phi) equals the surface area integral of B (or the magnetic flux). We separately consider the cases of a point within the solenoid and a point outside the solenoid. For the inside case the flux depends on s since flux = pi s^2 B. For the outside case the flux is constant at pi R^2 B. The two cases give expressions for A_phi(s) which are continuous at s = R. It is surprising that A(s > R) exists at all since the B-field disappears outside of the solenoid. Evidently A is of the form that A is not equal to zero but its curl is zero. From the expression for the curl in cylindrical coordinates you can see that if A is proportional to 1/s – which it is for s > R – the curl and B field will vanish. The A calculation was fairly simple but the B calculation from Ampere’s law was even simpler. Why bother with A at all?

The role of A in Lagrangian dynamics

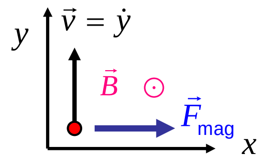
Advanced physics uses L rather than $\vec{F} = m\vec{a}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \text{ ; where } L\left(\{\dot{q}_\alpha, q_\alpha\}\right).$$

Usually $L = KE - PE = \frac{m}{2}\vec{v}\bullet\vec{v} - V(\vec{r})$. But for magnets we know $\vec{F} = q\vec{v}\times\vec{B}$. How do we get velocity dependent force into L?

Magnetic prescription is $V(\vec{r}) \to -q\vec{v}\bullet\vec{A}$. where $\vec{v} = (\dot{x} \quad \dot{y} \quad \dot{z})$.

$$\boxed{L = \frac{m}{2}\vec{v}\bullet\vec{v} + q\vec{v}\bullet\vec{A}}$$



Ex particle inside solenoid

$L = \frac{m}{2}\vec{v}\bullet\vec{v} + q\vec{v}\bullet\vec{A}$ where $\vec{v} = (\dot{x} \quad \dot{y} \quad \dot{z})$

and $\vec{A} = \frac{Bs}{2}\hat{\phi}$ where $\hat{\phi} = (-\sin\phi \quad \cos\phi)$

$$\vec{A} = \frac{B}{2}(-s\sin\phi \quad s\cos\phi) = \frac{B}{2}(-y \quad x)$$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{qB}{2}(x\dot{y} - y\dot{x})$$

$$\frac{\partial L}{\partial x} = \frac{qB\dot{y}}{2} \text{ ; } \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qBy}{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \to \frac{d}{dt}\left[m\dot{x} - \frac{qBy}{2}\right] = \frac{qB\dot{y}}{2}$$

$m\ddot{x} - \frac{qB\dot{y}}{2} = \frac{qB\dot{y}}{2} \to \boxed{m\ddot{x} = qB\dot{y}}$ check fig

How about outside of solenoid ($\vec{B} = 0$)?

$$\vec{A}(s > R) = \frac{BR^2\hat{\phi}}{2s} = \frac{BR^2(-y \quad x)}{2\left(x^2 + y^2\right)}$$

Is it true that $L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{qBR^2(x\dot{y} - y\dot{x})}{2\left(x^2 + y^2\right)}$

has solution $m\ddot{x} = m\ddot{y} = 0$???

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It turns out that A is the way magnetic forces are incorporated into modern theoretical mechanics (eg Lagrangian dynamics) and its quantum mechanical extension. The basic idea is that in Lagrangian dynamics the potential is king -- forces do not appear at all. You may be familiar with the Lagrangian written in the form KE – PE where the PE is just U(r-vec). This works fine for position dependent forces such as gravity or electrostatics – but what can we do about the velocity dependent Lorentz force? We need a scalar quantity that must be proportional to the charge, the velocity and a potential. The simplest, most natural thing to do would be to replace the potential with q v dot A. Astonishingly enough this is also the correct thing to do! To illustrate this we will compute the Eq. of motion for a charge in the uniform magnetic potential provided by the solenoid which we just obtained. We begin with the s<R case. Our first step is to write our q v dot B term in terms of r and rdot. We next need to get the equations of motion using Lagrange's Eq. As you might recall this involves comparing the partial derivative of L with respect to x or y and the total rate of change of the partial derivative of L with respect to xdot or ydot. As usual – a partial derivative is defined according to rules telling you what to hold constant. Here partial L/partial x this means hold y, xdot and ydot constant, and vice versa: partial L / partial xdot means hold y, x, and ydot constant. If we apply Lagrange's Eq. to this Lagrangian we get the equation for circular motion: m x doubledot = q B ydot which is shown in the figure. But what about the equation of motion outside of the solenoid? Clearly we must get zero acceleration since there is no magnetic field and thus no magnetic field. I ask you show this in homework – be prepared with a pot of coffee! Interestingly enough although the classical charge is blind to the A-field outside of a solenoid, the quantum charge is not! The outside A affects the energy levels in an experimentally verified way called the Aharonov-Bohm effect. Apparently the vector potential has a reality that transcends the magnetic field that it describes. The vector potential plays a much more important theoretical role than the scalar potential in mechanics, quantum mechanics, and electrodynamics. In Physics 436 you will learn that a modified form of our vector potential integral places a crucial role in the calculation of electromagnetic radiation from moving charges.

Multipole expansion of A

Recall P_ℓ generator function

$$\frac{1}{\mathfrak{r}} = \frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{|\vec{r}|} \sum_{\ell=0,1,2\cdots} \left(\frac{|\vec{r}\,'|}{|\vec{r}|}\right)^{\ell} P_\ell(\hat{r} \bullet \hat{r}\,')$$

We can combine this with $\vec{A}(r) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\ell}}{\mathfrak{r}}$

$$\vec{A}(r) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0,1,2,\cdots} \frac{1}{|\vec{r}|^{\ell+1}} \oint d\vec{\ell}\,'\, P_\ell(\hat{r} \bullet \hat{r}\,') |\vec{r}\,'|^{\ell}$$ Look at 1st terms

$$\vec{A}(r) \approx \frac{\mu_0 I}{4\pi} \left\{ \frac{\oint d\vec{\ell}}{r} + \frac{\oint d\vec{\ell}'\, \mathrm{r}'\, (\hat{r} \bullet \hat{r}\,')}{r^2} + \frac{\oint d\vec{\ell}'\, (\mathrm{r}')^2 \left[3(\hat{r} \bullet \hat{r}\,')^2 - 1\right]}{2r^3} + \cdots \right\}$$

$\oint d\vec{\ell} = 0$ so monopole term vanishes.

Lets compute dipole term for current loop of radius R.

$\oint d\vec{\ell}'\, \mathrm{r}'\, P_1 = \oint d\vec{\ell}\,'\; r'\; \hat{r} \bullet \hat{r}\,'$

For circular loop: $d\vec{\ell}\,' = R(-\sin\phi \quad \cos\phi \quad 0)\, d\phi$

$$\hat{r} = \frac{(x \quad y \quad z)}{\sqrt{x^2 + y^2 + z^2}} \; ; \; \hat{r}\,' = (\cos\phi \quad \sin\phi \quad 0)$$

$$\oint d\vec{\ell}\,'\; r'\; \hat{r} \bullet \hat{r}\,' = \oint d\vec{\ell}\,'\, \hat{r} \bullet \vec{r}\,'$$

$$= R^2 \int_0^{2\pi} \frac{(x\cos\phi' + y\sin\phi')}{\sqrt{x^2 + y^2 + z^2}} (-\sin\phi' \quad \cos\phi')\, d\phi'$$

$$\rightarrow \oint d\vec{\ell}\,'\; r'\; \hat{r} \bullet \hat{r}\,' = \frac{\pi R^2 (-y \quad x \quad 0)}{\sqrt{x^2 + y^2 + z^2}}$$

$$\vec{A}(r) = \frac{\mu_0 I \pi R^2 (-y \quad x)}{4\pi (x^2 + y^2 + z^2)^{3/2}} = \frac{\mu_0\; \vec{m} \times \hat{r}}{4\pi r^2}$$

with $\vec{m} = I\pi R^2 \hat{z}$; $\vec{r} = (x \quad y \quad z)$; $\hat{r} = \frac{\vec{r}}{r}$

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We conclude this long chapter with a discussion of the magnetic multipole expansion. This plays a critical role in understanding magnetic phenomena in materials in much the same way as the electric dipole played a critical role in polarization phenomena in materials. The main result is very easy to obtain because of the role of 1/|r-r'| as a generator function for the Legendre polynomials. We recycle this old identity that we first met in the Laplace Chapter. We basically have the expansion for A in one stroke. We next expand the first three terms. The lowest order (monopole) term involves the path integral of dL which vanishes as we saw earlier. Again this indicates that there are no magnetic monopoles. The lowest non-vanishing term is the magnetic dipole. This is easy to compute for a physical current loop of radius R which we place in the x-y plane. Again we begin by cleanly writing the vectors for the observation vector r and the source vector r'. Our line integral becomes an integral over phi'. Only terms proportional to sin^2(phi') or cos^2(phi') will survive the integration over phi'. We are left with an answer proportional to the current times the area of the circular loop which – of course– is the magnitude of the dipole moment. We agree with the simple cross product form obtained in Griffiths as long as the dipole moment m points perpendicular to the current loop in the z –direction according to the right-hand rule. On to magnetic materials!