Jim Wiss
Physics 435

Electromagnetic Fields and Lagrangians

We discuss how to insert electric and magnetic fields into the classical, non-relativistic Lagrangian of mechanics using the vector and scalar potentials. The prescription for a single charged particle is $L=\frac{1}{2}mv^2+q\vec{A}\bullet\vec{v}-qV$ where we are limiting our treatment to electrostatic and magnetostatic fields where $\vec{A}(\vec{r})$ and $V(\vec{r})$ and neither has an explicit time dependence.

A useful first example is a system with $\vec{B}=B_0\hat{z}$ and $\vec{A}=\frac{B_0 s}{2}\hat{\phi}=\frac{B_0}{2}\langle -y,x\rangle$ and $\vec{E}=-\vec{\nabla}V(x,y)$.

This has

$$L=\frac{1}{2}mv^2+q\vec{A}\bullet\vec{v}-qV=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{qB_0}{2}\left(-y\dot{x}+x\dot{y}\right)-qV(x,y)$$

Applying the Lagrange Equation for x, we have:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=\frac{\partial L}{\partial x}\ ;\ \frac{\partial L}{\partial \dot{x}}=m\dot{x}-\frac{qB_0 y}{2}\ ;\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=m\ddot{x}-\frac{qB_0}{2}\dot{y}\ ;\ \frac{\partial L}{\partial x}=\frac{qB_0\dot{y}}{2}-q\partial_x V$$

or $\rightarrow m\ddot{x}-\frac{qB_0}{2}\dot{y}=\frac{qB_0\dot{y}}{2}-q\partial_x V$ or $m\ddot{x}=qB_0\dot{y}-q\partial_x V=\vec{F}_x$ where $\vec{F}=q\vec{v}\times\vec{B}-q\vec{\nabla}V$

Our $\vec{A}=\frac{B_0 s}{2}\hat{\phi}$ expression was obtained using the flux method or $\oint\vec{A}\bullet d\vec{\ell}=\int_{<\vec{\ell}}\vec{B}\bullet d\vec{a}$ applied to a circular loop centered on the origin. This form naturally arises for a cylindrical magnet of radius R with $\vec{B}(s<R)=B_0\hat{z}$, but as long as we stay within the $B_0\hat{z}$ region, the centering of the $\oint\vec{A}\bullet d\vec{\ell}$ loop or the symmetry of the pole pieces cannot affect the equation of motion which only depends on the $B_0\hat{z}$ magnetic field. Gauge Transformations of the form $\vec{A}'=\vec{A}+\vec{\nabla}\varphi$ allows us to change $\vec{A}$ while preserving $\vec{B}$. Since the motion of a charge in a static $\vec{E}$ and $\vec{B}$ field can only depend on $\vec{F}=q\vec{v}\times\vec{B}-q\vec{\nabla}V$, evidently the Lagrange formalism must be blind to the changes in L due to a gauge transformation $\varphi(\vec{r})$.

Let's illustrate this for a charge particle moving in a magnetic field with $L=\frac{1}{2}mv^2+q\vec{A}\bullet\vec{v}$.

Let $L=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{qB_0}{2}\vec{v}\bullet\vec{A}$ and $L'=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{qB_0}{2}\vec{v}\bullet\left(\vec{A}+\vec{\nabla}\varphi\right)$. Applying Lagrange's equation for L' we have:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = \frac{\partial L'}{\partial x} \; ; \; \frac{\partial L'}{\partial \dot{x}} = m\dot{x} - qA_x + q\partial_x \varphi \; ; \; \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = m\ddot{x} + q\dot{A}_x + q\partial_x \dot{\varphi} \; ;$$
$$\frac{\partial L'}{\partial x} = q\left(\dot{x}\partial_x A_x + \dot{y}\partial_x A_y\right) + q\left(\dot{x}\partial_X^2 \varphi + \dot{y}\partial_Y \partial_x \varphi\right)$$

Let's now work with φ. Since $\varphi(x,y)$, we only get φ through $\dot{x}$ and $\dot{y}$ using $d\varphi = dx\partial_x\varphi + dy\partial_x\varphi \rightarrow \dot{\varphi} = \dot{x}\partial_x\varphi + \dot{y}\partial_x\varphi$. Thus $\partial_x\dot{\varphi} = \dot{x}\partial_x^2\varphi + \dot{y}\partial_y\partial_x\varphi$. Assembling the pieces we have

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = \frac{\partial L'}{\partial x} \; ; \; \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = m\ddot{x} + q\dot{A}_x + q\left(\dot{x}\partial_x^2\varphi + \dot{y}\partial_y\partial_x\varphi\right) \; ; \; \frac{\partial L'}{\partial x} = q\left(\dot{x}\partial_x A_x + \dot{y}\partial_x A_y\right) + q\left(\dot{x}\partial_X^2 \varphi + \dot{y}\partial_Y \partial_x \varphi\right)$$
$$\rightarrow m\ddot{x} + q\dot{A}_x = q\left(\dot{x}\partial_x A_x + \dot{y}\partial_x A_y\right)$$

which is independent of the gauge function φ. Hence L indeed has gauge freedom since $\vec{A}' = \vec{A} + \vec{\nabla}\varphi$ produces the same equation of motion as $\vec{A}$.

Some $\vec{A} = \frac{B_0 s}{2}\hat{\phi}$ examples.

Re-centering the origin to x_0, y_0 gives $\vec{A}' = \vec{A} + \frac{B_0}{2}\left(-y_0\hat{x} + x_0\hat{y}\right) \rightarrow \varphi = \frac{B_0}{2}\left(-y_0 x + x_0 y\right)$ which won't change the equation of motion due to gauge freedom.

Another example uses $\varphi = \frac{B_0}{2}xy$ which changes $\vec{A} = \frac{B_0 s}{2}\hat{\phi} = \frac{B_0}{2}\left\langle -y, x\right\rangle$ to $\vec{A}' = B_0 x\hat{y}$ which no longer has cylindrical symmetry. You can easily confirm using a determinant that $\vec{A}'$ preserves $\vec{B} = B_0\hat{z}$ as well as the equation of motion.