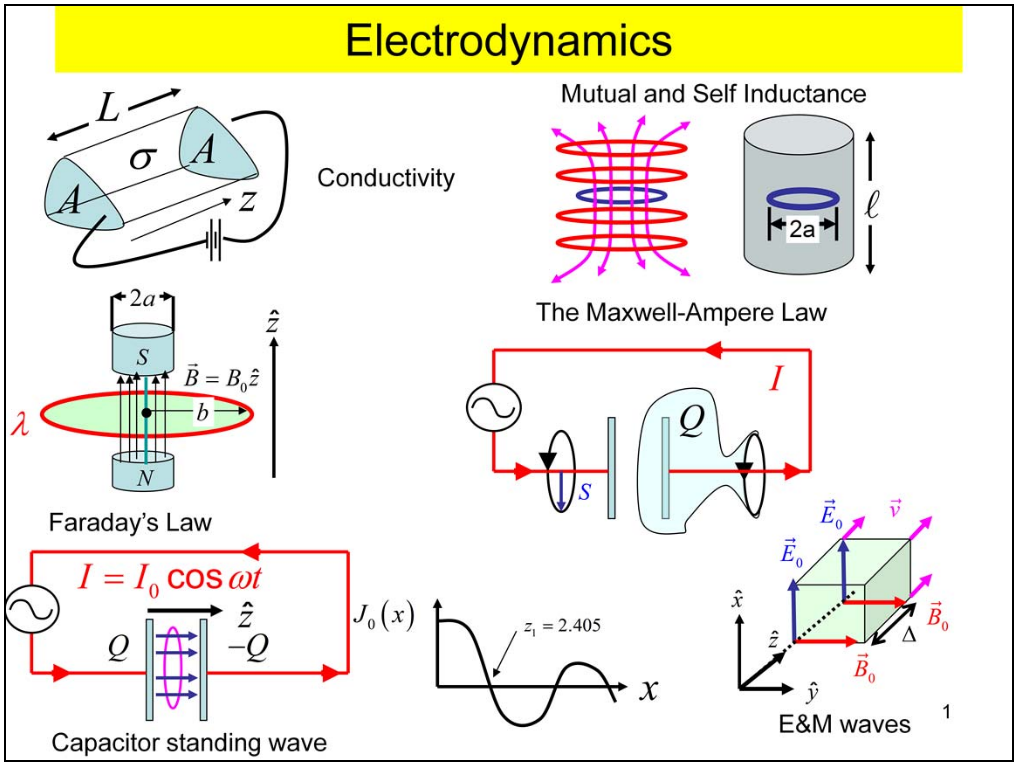
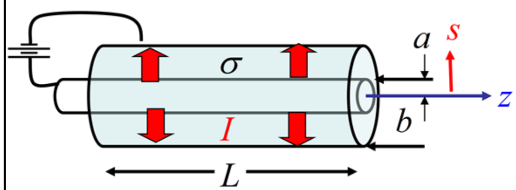


We put the finishing touches on the complete set of Maxwell's Equations in this Chapter. This is our first chapter on changing rather than static electric and magnetic fields. We begin with a discussion of conductivity which allows us to compute the resistance for arbitrarily shaped conductors. We show that the if the conductivity of a material is uniform, the potential in a resistive medium obeys Laplace's Equation. This allows us to obtain an interesting and unexpected relationship between the capacitance and the resistance between two arbitrarily shaped conductors. We next discuss Faraday's Law. We show some of the physics of Faraday's law is nothing more than the magnetic force law but there is some new physics in the case where nothing moves in a circuit but the magnetic field changes in time. We work an interesting example where a collapsing magnetic field causes a disk to acquire angular momentum. Presumably this angular momentum must have resided in the original magnetic field. We next consider mutual and self inductance and prove the interesting but implausible result that the flux to current ratio for the flux loop A due to a current in loop B is the same as that flux to current ratio for the flux in loop B due to a current in loop A. This is independent of the geometry of the two loops. We next show how Maxwell fixed a basic inconsistency in Ampere's law that violated conservation of charge. Maxwell showed that a changing electric field can create a magnetic field in the absence of physical currents. In Physics 212 the extra term introduced by Maxwell was called the displacement current. We next discuss the standing wave set up by an AC current flowing through a capacitor. We develop an infinite series in increasing powers of the AC frequency driving the capacitor. We give a preview of our 436 treatment of EM waves using a very simple-to-calculate “slab” wave which illustrates how EM fields can travel in a vacuum with no current or charges. We conclude with a discussion of how the Ampere-Maxwell Law must be modified in materials.

Resistance

In many materials $\vec{J} = \sigma \vec{E}$ where $\sigma = 1/\rho$ is conductivity.
This is not fundamental but a complicated effect.
A related concept is resistance $R \propto \rho$ where $\Delta V = IR$
We can calculate R for novel situations. We start with a long cylinder.



Our principle will be conservation of I.
For $L \gg a,b$ J only depends on s

$$I = \int_S \vec{J} \cdot d\vec{a} = 2\pi s L\, J = 2\pi s L\, \sigma |E_s|$$

$$\vec{E} = \frac{I}{2\pi s L\, \sigma}\hat{s} \; ; \; \Delta V = -\int_b^a \vec{E} \cdot d\vec{\ell} = \frac{I}{2\pi L\, \sigma}\int_a^b \frac{ds}{s}$$

$$\Delta V = \frac{I}{2\pi L\, \sigma}\ln\frac{b}{a} \to R = \frac{\Delta V}{I} = \frac{\ln(b/a)}{2\pi L\, \sigma}$$

We note our E-field looks the same as electrostatic E-field

$$\vec{E} = \frac{I}{2\pi s L\, \sigma}\hat{s} \;\to\; \vec{E} = \frac{Q/L}{2\pi\varepsilon_0 s}\hat{s}$$

under substitution $I = \dfrac{\sigma Q}{\varepsilon_0}$ Why?

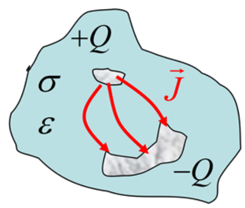
$$\vec{J} = \sigma \vec{E} \; ; \; I = \int_S \vec{J} \cdot d\vec{a} = \int_S \sigma \vec{E} \cdot d\vec{a} = \sigma \int_S \vec{E} \cdot d\vec{a}$$

(assuming σ is constant)

But $\int_S \vec{E} \cdot d\vec{a} = \dfrac{Q}{\varepsilon_0} \;\to I = \dfrac{\sigma Q}{\varepsilon_0}$

This relates R and C for arb geometry

$$R = \frac{\Delta V}{I}, \; \frac{1}{C} = \frac{\Delta V}{Q} \to R = \frac{\Delta V}{[\sigma Q/\varepsilon_0]_I} = \frac{\varepsilon_0}{\sigma C}$$



Discharge time
$Q = Q_0 \exp(-t/\tau)$
$\tau = RC = \dfrac{\varepsilon}{\sigma}$
For all geometries

2

The “Ohm’s Law” relationship between current and voltage or (in this case) E-field and J current density is not fundamental and depends on some complicated transport properties of electrons through conductors. It isn’t always true but often true. We can use the J = sigma E form to compute the resistance for various geometries. Its unfortunate that sigma is used for surface charge density and conductivity. Of course we could always use 1/rho where rho is the resistivity but rho is used for volume charge density! As our first example we consider a long cylinder with conductivity sigma with inner radius “a” and outer radius “b”. A battery sets up a voltage between the inner electrode at s = a and the outer electrode at s = b and a current flows through the system. At this point all we know is J = sigma E but this is enough to solve the problem if we assume a static situation such that charge is not piling up in the resistive material. From symmetry we can argue that J cannot depend on phi. In the limit of infinite L, there can be no z dependence either. Hence J can only depend on the cylindrical coordinate s. For a given s, J passes through a transverse area of 2 pi s L and thus the current passing through this area is I = 2 pi s L J. Since no charge is piling up (steady state) I is constant and thus J = I/(2 pi s L). Since J = sigma E, E has the same 1/s form as long as we assume a uniform conductivity sigma. We can then find the voltage difference between radius a and b by integrating E ds. The voltage over the current is the resistance. I find it interesting that the E field in the resistive cylinder looks very similar to the E-field due to a cylinder of charge that we can easily get from Gauss’s law. The Gauss’s law field turns into the Ohm’s law field by under the indicated current to charge substitution. If we apply this same current to charge substitution to the definition of resistance as voltage over current we get voltage/charge which proportional to 1/C. Within some constants evidently 1/R and C are basically the same thing! In fact its very easy to verify the current to voltage substitution for the case of a uniform (constant) sigma. We write I as the surface integral of J dot area. Since J equals sigma E, we have the current is proportional to the surface integral of E dot area. But according to Gauss’s law this integral is proportional to the enclosed charge Q. So this current to charge substitution is not just a mathematical stunt. The charge on the inner conductor is really proportional to the current through this cylindrical resistor. Our Gauss’s Law argument implies that RC is epsilon/sigma **independent** of the geometry. Recall from Physics 212 that RC is the capacitor discharge time constant. Hence the discharge time constant is epsilon/sigma for an arbitrary geometry of electrodes inside of a conductive medium. A fairly remarkable result!

Resisters and Laplace's Law

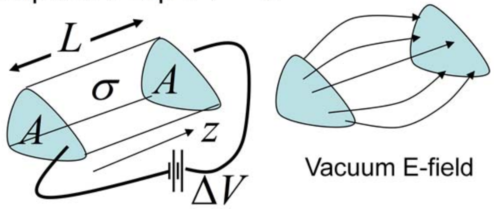
$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0\rightarrow\vec{\nabla}\cdot\vec{J}=0\rightarrow\vec{\nabla}\cdot\left(\sigma\vec{E}\right)=0$$

$$\vec{\nabla}\cdot\left(\sigma\vec{E}\right)=\boxed{\sigma\;\vec{\nabla}\cdot\vec{E}+\vec{E}\cdot\vec{\nabla}\sigma=0}$$

If $\vec{\nabla}\sigma=0\rightarrow\vec{\nabla}\cdot\vec{E}=0\;\;\rightarrow\vec{\nabla}\cdot\vec{\nabla}V=0$

Hence uniform $\sigma\rightarrow$ V satisfies

Laplace's Eq. $\nabla^2V=0$



We can use $\nabla^2V=0$ to find the field in a quasi cylinder . The BC are V=0 at 0 and V = ΔV at $z=L$

$V\left(z\right)=\frac{\Delta\mathrm{V}}{L}z$ solves Laplace with BC

Hence $\vec{E}\;=-\hat{z}\frac{\partial}{\partial z}\left(\frac{\Delta V}{L}z\right)=-\frac{\Delta V}{L}\hat{z}$

$$\vec{J}=\sigma\vec{E}=-\sigma\frac{\Delta V}{L}\hat{z}$$

$$I=\int_S\vec{J}\cdot d\vec{a}=\sigma\frac{\Delta V}{L}A$$

$$I=\frac{\Delta V}{R}\rightarrow\boxed{R=\frac{L}{A\sigma}}$$

What if $\sigma(\vec{r})$? $\vec{\nabla}\sigma\neq0$

$$\boxed{\sigma\;\vec{\nabla}\cdot\vec{E}+\vec{E}\cdot\vec{\nabla}\sigma=0}$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}\rightarrow\boxed{\rho=-\frac{\varepsilon_0\vec{E}\cdot\vec{\nabla}\sigma}{\sigma}}$$

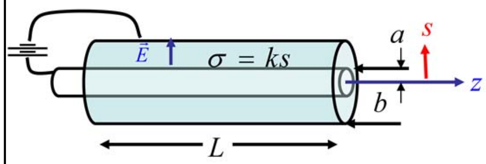
Hence a non-uniform σ will create a volume charge when passing a steady current.

$\rightarrow\nabla^2V\neq0$

3

Often the electrical fields in resistive problems satisfy the Laplace Eq. We begin with the continuity equation encapsulates conservation of charge. If we assume that we have a steady flow of current, there will be no time dependent change in the charge density rho. This means the current density J has no divergence. If J = sigma E this means the divergence of sigma times E is zero. We use one of the product rules from Griffiths and write the divergence of sigma times E in terms of sigma times the divergence of E and E dot the gradient of sigma. If sigma is uniform (independent of position) the gradient of sigma will be zero and we can conclude that the gradient of E is zero. This means there is no volume charge density and if we write the E-field as negative the gradient of a potential V, this potential satisfies Laplace's Eq. This observation greatly simplifies many resistance problems. Here is an example, Let us assume that we have "cylinder" with a uniform conductivity with an irregularly shaped ends. We attach perfectly conducting electrodes to the two ends and establish a current with a battery. Since the conductivity is uniform, our potential satisfies Laplace's Eq. The boundary conditions are the two ends, of area A, are equipotentials and there is a voltage difference of Delta V between these two equipotentials which are separated by a distance L. We take the separation to be in the z direction. If we assume V = delta V z /L over the full area A we satisfy the constant V BC on the end plates as well as Laplace's Eq. and have a constant E-field in between the plates. But as L becomes large compared to the dimensions of the end plate, one would expect to find considerable fringe fields if the two plates were in a vacuum as illustrated on the right. Evidently we don't have the unique solution from just the end plate boundary conditions. This is because we haven't specified the potential on all parts of a bounding surface which was the condition for the uniqueness theorem that we proved in the Laplace chapter. There is a second uniqueness theorem proved in Griffiths which says that one can specify either the potential or gradient of the potential on all regions of the bounding surface to guarantee a unique solution. For the case of a conducting medium, we cannot have J with a component in the s-hat direction or we would have current flowing out of the cylinder into vacuum. This additional condition does not apply to the E-field from two plates in a vacuum and hence the potential and field would be much more complicated because of the fringe fields on the plates. Here we get a uniform E-field over the full area A. We next convert this to a uniform current density J and a current J times A. We end up with a very simple expression for the resistance R. The resistance is proportional to the length – which suggest resisters in series (longer) just add and resisters in parallel (more area) add in inverse.

Non-uniform conductivity example



Redo the cylinder assuming $\sigma = ks$

Again we solve this using constant I.

For L $\gg$ a,b J only depends on s

$$I = \int_S \vec{J}\cdot d\vec{a} = 2\pi sL\ J = 2\pi sL\ \sigma E_s = 2\pi ks^2 LE_s$$

$$\rightarrow \vec{E} = -\frac{I}{2\pi ks^2 L}$$; Note $\vec{E}$ is not $\propto \frac{\hat{s}}{s}$

$$\Delta V = -\int_b^a \vec{E}\cdot d\vec{\ell} = \frac{I}{2\pi L\ k}\int_a^b \frac{ds}{s^2}$$

Find R: $\Delta V = \frac{I}{2\pi L\ k}\left(\frac{1}{a} - \frac{1}{b}\right)$

$$\rightarrow R = \frac{\Delta V}{I} = \frac{1}{2\pi L\ k}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Check $\rho = -\frac{\varepsilon_0 \vec{E}\cdot\vec{\nabla}\sigma}{\sigma}$

$$\vec{E} = \frac{I}{2\pi ks^2 L}\hat{s}\ ;\ \vec{\nabla}\sigma = \hat{s}\frac{\partial ks}{\partial s} = k\hat{s}\ ;$$

$$-\frac{\varepsilon_0 \vec{E}\cdot\vec{\nabla}\sigma}{\sigma} = -\frac{\varepsilon_0 I\hat{s}\cdot\left(k\hat{s}\right)_{\nabla\sigma}}{2\pi ks^2 L\left(ks\right)_{\sigma}} = -\frac{\varepsilon_0 I}{2\pi ks^3 L}$$

We next calculate ρ using $\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho$

$$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \frac{\varepsilon_0 I}{2\pi kL}\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{s}{s^2}\right) = -\frac{\varepsilon_0 I}{2\pi kLs^3} = \rho$$

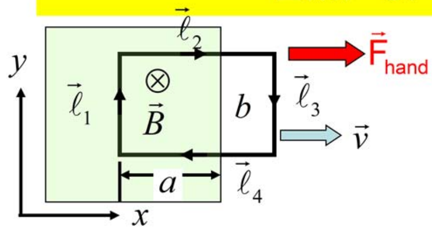
$\rightarrow$ agrees w/ $\boxed{\rho = -\frac{\varepsilon_0 \vec{E}\cdot\vec{\nabla}\sigma}{\sigma}}$

Since $\rho \neq 0$ it is not surprising that V does not satisfy the Laplace Eq.

4

We now consider a case where the conductivity is proportional to s (the distance from the cylinder axis). We expect to find a bound charge density in the conductive medium and hence will not be able to use Laplace's Eq. We can rely on the fact that we have a constant current. For a long cylinder, we can count on the fact that the current density J can only depend on s. This is because in the infinite L limit, every z spot looks the same so there can be no dependence on z. All azimuths look the same because of the rotational symmetry. This means J can only depend on s. The total current I is then J times A where A is the cylinder area. Setting the E-field times the s-dependent conductivity gives J and J times 2 pi s L equals the constant current. We end up with an E-field that is inversely proportional to s^2 where an ordinary charged cylinder or the constant conductivity cylinder as E inversely proportional to s. Hence we don't have the Laplace solution of an ordinary cylinder. We can get the voltage difference between the inner and outer cylinder by integrating the E-field and obtain a voltage related to the inverse of "a" minus the inverse of "b" where as the ordinary cylinder voltage involves the log of b/a. The voltage over the current then gives a new form for the resistance. We can then get the charge volume density by taking the divergence of our E-field in cylindrical coordinates. We get a charge density that is proportional to the current and if we evaluated my form involving the gradient of the conductivity we would have the same answer. Although the volume charge in the conductive medium is proportional to current, you don't want to think of this charge somehow piling up in higher resistance regions. This is after all steady flow with a constant current and time independent charge density. Rather the change that an E-field undergoes so it can produce the same current as the conductivity changes must be accompanied by charge in order to be consistent with Gauss's law.

EMF of a moving wire



$\vec{F} = q\vec{v}\times\vec{B} \rightarrow \vec{f} = \frac{\vec{F}}{q}\vec{v}\times\vec{B}$; $\vec{f} \Leftrightarrow \vec{E}$

$\vec{B} = -B\hat{z}$; $\vec{v} = v\hat{x}$; $\vec{f} = vB\hat{y}$ for 1,2,4

Define EMF = $\varepsilon = \oint \vec{f}\cdot d\vec{\ell}$

$\varepsilon = \sum_{i=1,4} \vec{f}_i\cdot\vec{\ell}_i = f_1\ell_1 = vBb$

Define $\Phi_m = \vec{B}\cdot(ab\hat{\eta}) = \vec{B}\cdot(-ab\hat{z}) = abB$

We use RH-rule to set $\hat{\eta}$

$-\frac{d\Phi_m}{dt} = -\frac{dabB}{dt} = -bB\frac{da}{dt} = -bB(-v) = \varepsilon$

$\rightarrow \boxed{\varepsilon = -\frac{d\Phi_m}{dt}}$ Faraday's Law

Add some resistance R to get current

$\varepsilon = vBb \rightarrow I = \frac{vBb}{R}$

$\vec{F}_{mag} = I\int d\vec{\ell}\times\vec{B} = -\frac{vB^2b^2}{R}\hat{x}$

Power = $\vec{F}_{hand}\cdot\vec{v} = \left(\frac{vB^2b^2}{R}\hat{x}\right)\cdot v\hat{x} = \frac{v^2B^2b^2}{R}$

$\text{Power}_{heat} = I^2R = \left(\frac{vBb}{R}\right)^2 R = \text{Power}$

Lenz's Law: Note induced current in direction to counter change in flux. I is clockwise thus induced B is in external B direction and tries to restore diminishing flux.

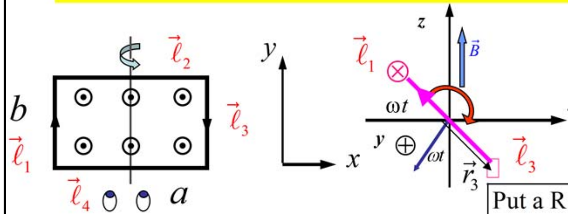
Lenz's Law is automatically satisfied w/ our RHR area vector convention.

Apparently Faraday's law is nothing more than the Lorentz force??

5

Griffiths does a good job at explaining what's old and what's new about Faraday's law concerning the emf created by a changing flux through a current loop. We start our discussion with the hopefully familiar example of a current loop being pulled out of constant magnetic field region such as the pole of a magnet with velocity v. The Faraday's Law treatment would be the magnetic flux through the loop is decreasing which induces an emf that drives a current. An alternative view is that the free charges in the wire loop pick up a velocity v in the z direction. They thus (on average) feel a magnetic force given by the Lorentz force expression of F = q v cross B. Only in the leftmost segment (labeled L_1) will the charges will experience work due to this magnetic force. The other segments are either out of the B-field (L_3) or have the magnetic force transverse to the velocity. We follow Griffith's notation of discussing the little f which is the force per unit charge or F/q. f is essentially the E-field but we will keep a distinction for a while since other E-fields might be present as well. We sum up the f_i dot dL_i contribution (only #1 contributes) to get the work per unit charge. For induction problems such as this we call this the emf=electromotive force which in this case is vBb and is positive for the path illustrated with arrows. We show next that this is the negative of the rate of change of flux through the loop. The flux is the integral of the B-field dotted into the area element. But what defines the direction of the area element? For the case of Gauss's law the area points out of the surface surrounding the charge. For the magnetic flux in the context of Faraday's law the area normal (eta-hat) is given by the right-hand rule – your thumb points along eta-hat when your fingers curl in the "guessed" direction of the current (in this case our guess given by the current arrows). The emf acts like a battery and will drive a current through the loop. This current will then apply a force on the wire which requires a "hand" to move the wire. If we calculate the power the hand supplies to the wire we find that it is I^2 R which is the power dissipated by the resistor in the form of heat. Don't get confused, the magnetic field does no work, the hand does, and the work that the hand does appears as heat.

EMF of a rotating loop



$$\vec{r}_1 \bullet \hat{z} = \frac{a}{2}\cos\left(\frac{\pi}{2} - \omega t\right)$$

$$\vec{f} = \vec{v} \times \vec{B} = \left(\vec{\omega} \times \vec{r}\right) \times \vec{B} = -\vec{\omega}\left(\vec{r} \bullet \vec{B}\right) + \vec{r}\left(\vec{\omega} \bullet \vec{B}\right)$$

Since $\vec{\omega} = \omega\hat{y}$ and $\vec{B} = B\hat{z} \rightarrow \vec{\omega} \bullet \vec{B} = 0$

$$\varepsilon = \sum_i \vec{f}_i \bullet \vec{\ell}_i = \sum_{i=1,4} -\vec{\omega} \bullet \vec{\ell}_i \left(\vec{r}_i \bullet \vec{B}\right) = -\omega B \sum_{i=1,3} \hat{y} \bullet \vec{\ell}_i \left(\vec{r}_i \bullet \hat{z}\right)$$

Where $\vec{r}_i$ to $\vec{\ell}_i$ center. $\hat{y} \bullet \vec{\ell}_1\left(\vec{r}_1 \bullet \hat{z}\right) = \hat{y} \bullet \vec{\ell}_3\left(\vec{r}_3 \bullet \hat{z}\right)$

$$\hat{y} \bullet \vec{\ell}_1\left(\vec{r}_1 \bullet \hat{z}\right) = b\left(\frac{a}{2}\sin\omega t\right) \rightarrow \varepsilon = -\omega Bab\sin\omega t$$

$$\Phi_m = \vec{B} \bullet \vec{A} = \vec{B} \bullet \left(ab\hat{\eta}\right) = -abB\cos\omega t$$

$$\varepsilon = -\frac{\partial \Phi_m}{\partial t} = -abB\omega\sin\omega t \text{ (it checks)}$$

But again no physics beyond magnetic force

Put a R resister in circuit

$$I = \frac{\varepsilon}{R} = \frac{-abB\omega}{R}\sin\omega t$$

$$\vec{m} = Iab\hat{\eta} = \frac{-a^2b^2B\omega\sin\omega t}{R}\left(-\sin\omega t \quad 0 \quad -\cos\omega t\right)$$

$$\vec{N} = \vec{m} \times \vec{B} = \frac{-a^2b^2B\omega\sin\omega t}{R}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\omega t & 0 & \cos\omega t \\ 0 & 0 & B \end{vmatrix}$$

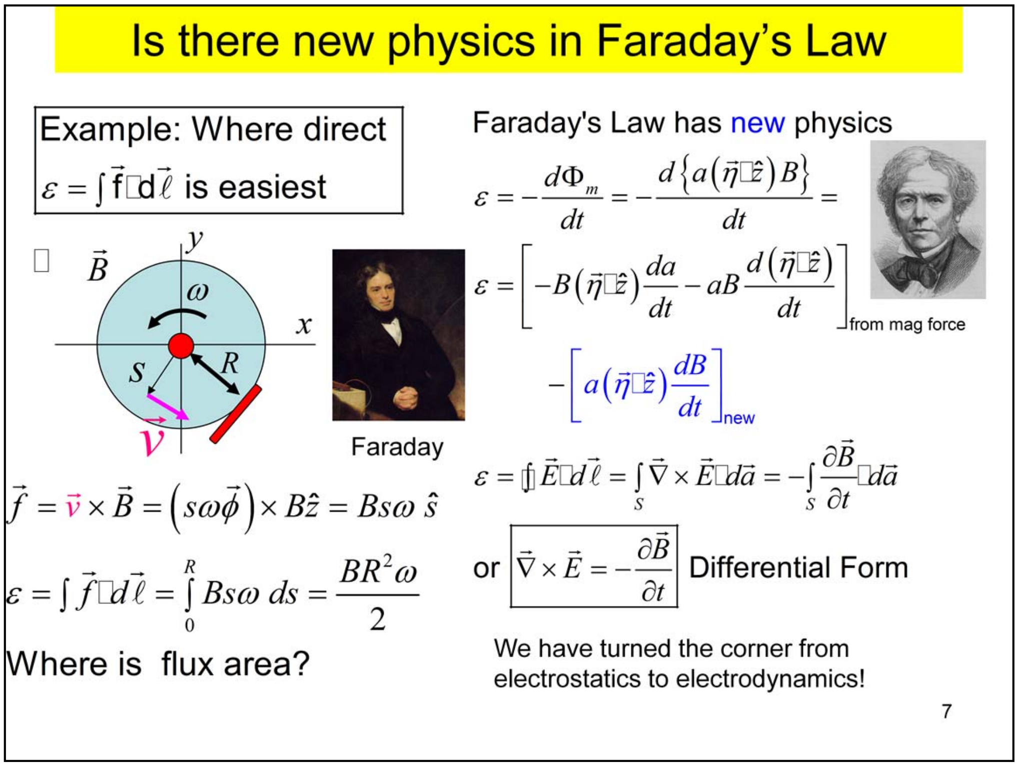
$$\vec{N} = \frac{-a^2b^2B^2\omega\sin^2\omega t}{R}\hat{y}$$

$$\rightarrow \vec{N}_{motor} = \frac{a^2b^2B^2\omega\sin^2\omega t}{R}\hat{y}$$

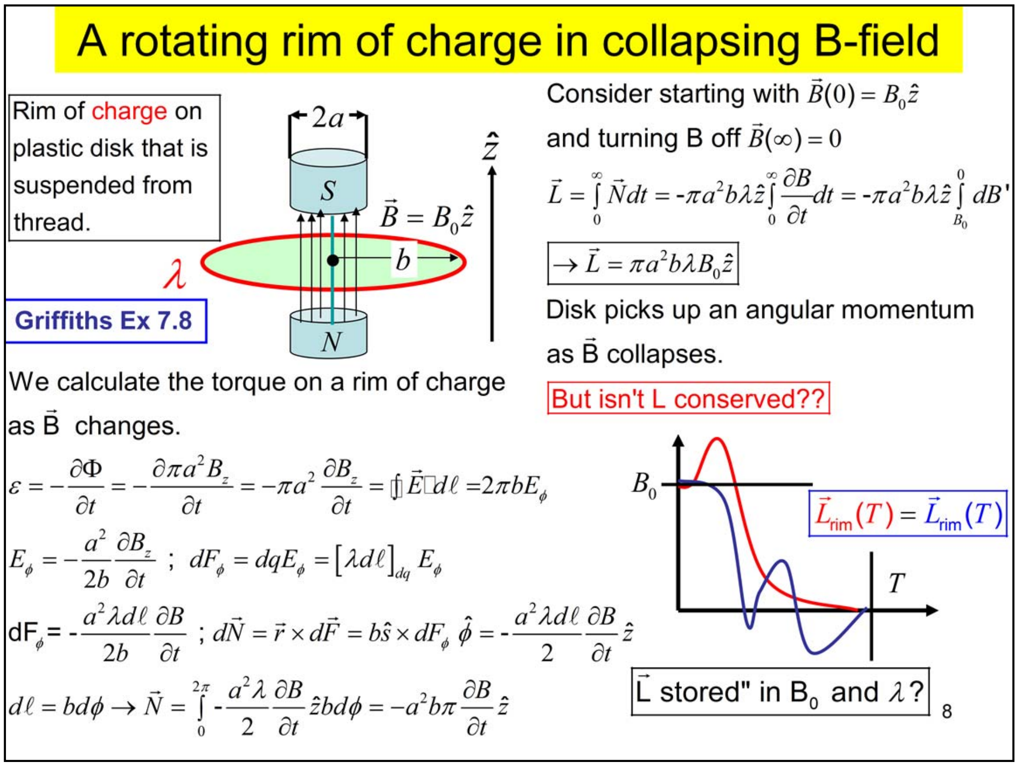
$$\text{power} = \vec{N} \bullet \vec{\omega} = \frac{a^2b^2B^2\omega^2\sin^2\omega t}{R} = I^2R$$

6

As you probably recall, you can also generate an emf by rotating a loop in a uniform magnetic field. This is the method of electrical energy production by a generator. We will show that this emf can also be calculated from Faraday's Law and show that power required to rotate the loop is equal to the power dissipation through a resistor. Again our force per unit charge is given by v cross B. Now we have a rotational rather than translational velocity which is given by omega cross r. Here r is the location of a charge. The omega is along the y-axis, the B-field is in the z direction. We could certainly evaluate the (omega cross r) cross B triple product by components but it is slightly easier if you use the so called bac-cab identity in Griffiths Chp. 1. The bac part vanishes since omega is transverse to B leaving a single term which implies f is parallel-or-antiparallel to omega. This means only segments #1 and #3 can contribute to the emf since segments #2 and #4 are directed transverse to omega. A little reflection will convince you that all points on segment #1 and #3 move with the same velocity so we can take r with respect to the segment center for #1 and #3. Since the v and L of segment #3 reverse with respect to those of segment 1, we get equal emf contributions and can find the emf of segment #1 and double it. We are left with two simple dot products to evaluate to get our emf expression. The r1 dot z-hat involves the cosine of pi/2-omega t. We next check the emf expression that we would get from the negative of the rate of change of flux. We use the same direction as for our line segments as used for the emf calculation. By the right-hand rule eta-hat points as shown in the rightmost figure. Our flux is abB cos(omega t) with this convention. We differentiate it with respect to time and get the same emf expression is from our f calculation. Again, the use of Faraday's law for a rotating loop contains no physics beyond f = v cross B. Just for fun we give the loop some resistance so we can calculate a current (again in our guessed direction given by the arrows). We are interested in the torque N that the B-field creates when interacting with the induced current. So we first compute the magnetic moment and then cross it with B-field to get the torque. Interestingly enough the torque is against the y-axis. The induced current is requiring that the generator does work to turn the coil and this torque direction never changes (although its magnitude does). Just like the power equals the force dotted into the velocity it also equals the torque dotted into the angular velocity, omega. We again find that we must supply a power equal to the dissipated power through the resister.



We conclude with an interesting example from Griffiths. In this case we rotate a conductive disk normal to a field with an angular velocity omega. We look at the emf between the axle and the rim. Again we can easily compute the force per unit charge by f = v cross B with B in the z-direction and v in the phi-direction. The emf is in the s-hat direction and varies linearly in s. We can easily integrate f dot dL to get an emf expression. It is difficult to imagine trying to do this via emf is negative d Phi / dt since there is no obvious circuit loop and therefore no obvious flux calculation. Before you get the impression that Faraday's law is just an application of the magnetic force law and contains no new physics, let us consider the three ways of getting a flux change. Since flux is B times A times the cosine between them, we can use the product rule of differentiation to get the emf contributions from changing flux area, changing the cosine by rotation, and changing the B-field. I showed you that the first two contributions are nothing more than the force law, but the third –getting an emf by changing B in a static loop-- is actual new physics. Faraday is bringing something new to the party. We can easily use the Stokes' theorem to relate our emf as a the rate of change of a flux integral to a more elegant expression which says the curl of the E-field is the negative of the rate of change of the B-field. We will call this Faraday's law. It is really our first expression that describes directly how a changing B-field can create an E-field. The new physics due to rate of change means we aren't discussing electrostatics or magnetostatics but rather electrodynamics.



In this example we consider an interesting use of Faraday’s Law which seems somewhat counterintuitive. We have an charge (with linear charge density lambda) on the rim of an insulating plastic disk of radius “b” > “a” which lies outside a uniform magnetic field B (perhaps from a a long solenoid with a small gap). The disk is free to rotate around a frictionless pivot held from thread but is initially at rest. A change in B-field will create an emf at radius “b” according to Faraday’s law. The flux through the disk is pi b^2 B_z and if B changes the rate of flux change will be pi a^2 times dB_z/dt. Since we evidently defined the area vector in the z-hat direction since our flux is proportional to B_z, our line integral of the E-field is just 2 pi b E_phi. It is interesting to note that we have an E_phi on the rim even though there is no B-field there. The integral of the force on the rim will vanish since the forces add in a circle to zero, but since all of the charge is on the rim it is easy to compute the torque on the rim. Each torque element is r cross dF. The force element dF = E dq. Since lambda is the force per unit length, dq = Lambda dL where dL is an element of the path along the rim. Thus dF = E lambda d L. An element of torque dN is r cross dF where r is in the s-hat direction and dF is in the phi_hat direction. The cross product is in z-hat direction for each charge on the rim. We can write dL = b dphi and do the simple integral to get the total torque in term of the rate of change of the B-field. Since the torque is the rate of change of angular momentum we can get the change in angular momentum by the time integral of the torque. An interesting case to consider is the disk is at rest and the B-field is on. We then collapse the field to zero. Our integral over time can be easily converted to an integral over dB to give us a simple expression for the angular momentum of the disk. The red path angular momentum equals the blue path angular momentum. L only depends on B_0 and lambda and is independent of the time evolution of the collapsing field. But if angular momentum is conserved, where did the angular momentum of the disk come from? Perhaps it came from the field itself and is intrinsic to the field rather than how the field is collapsed.

Mutual Inductance

Faraday's law says a changing current in one loop can induce an emf in another. This is principle of transformers. This is quantified by mutual inductance.

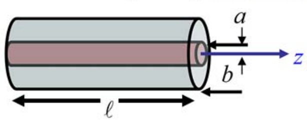
$$\Phi_2 = M_{12} I_1 \;\; ; \Phi_2 = \int_S \vec{B} \bullet d a_2 = \int_S \nabla \times \vec{A} \bullet d a_2 = \oint \vec{A}(\vec{r}_2) \bullet d\vec{\ell}_2$$

A is due to loop 1 $\vec{A}(\vec{r}_2) = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{\ell}_1}{|\vec{r}_2 - \vec{r}_1|}$

$$\Phi_2 = \oint \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{\ell}_1}{|\vec{r}_2 - \vec{r}_1|} \bullet d\vec{\ell}_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{\ell}_1 \bullet d\vec{\ell}_2}{|\vec{r}_2 - \vec{r}_1|}$$

$M_{12} = \frac{\Phi_2}{I_1} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{\ell}_1 \bullet d\vec{\ell}_2}{|\vec{r}_2 - \vec{r}_1|}$ (Neumann Formula)

As you can see $M_{12} = M_{21}$ Hard to believe!

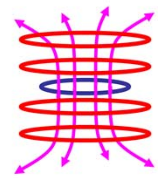


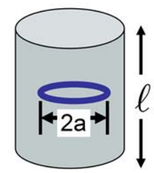
$$\Phi_b = M_{ba} I_a \;\; ; \Phi_b = N_b \pi a^2 B_a = \pi a^2 N_b \left(\frac{\mu_0 N_a I_a}{\ell} \right)$$

$$M_{ba} = \frac{\mu_0 N_a N_b}{\ell} \pi a^2$$

$$\Phi_a = M_{ab} I_b \;\; ; \Phi_a = N_a \pi a^2 B_b = \pi a^2 N_a \left(\frac{\mu_0 N_b I_b}{\ell} \right)$$

$M_{ab} = \frac{\mu_0 N_a N_b}{\ell} \pi a^2 = M_{ba}$ This is easy to believe





Mutual inductance from single loop

$M_{ab} = \frac{\Phi_a}{I_b}$; $\Phi_a = \frac{\mu_0 I_b N_b}{\ell} \pi a^2$

$M_{ab} = \frac{\mu_0 N_b}{\ell} \pi a^2 = \frac{\Phi_b}{I_a}$

Is it true $\Phi_b = \frac{\mu_0 I_a N_b}{\ell} \pi a^2$??

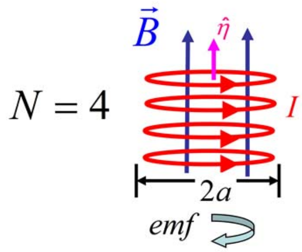
Hard to believe given complicated fields

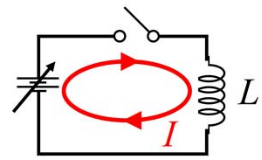
Strange but true!

9

If an oscillating magnetic field from loop 1, can enter loop 2, it will induce an emf in loop 2 according to Faraday's law. The usual way this is done is to provide an oscillating current in loop 2. One important example is a transformer, One circuit contains a solenoid (loop 1) which is wrapped around a solenoid in an independent circuit (loop 2) so the magnetic field of the (loop 1) provides a changing flux in loop 2—thus creating an emf in loop 2. We can quantify this process by the mutual inductance M_12 which relates the magnetic flux in the first circuit to the current in the second circuit. By Faraday's law, the rate of change of current in loop 1 times the mutual inductance will be the emf created in loop 2. We can chain together some of our magnetic tricks to write a compact expression for the mutual inductance called the Neumann formula. This trick uses Stokes' Theorem to write the magnetic flux in loop 2 which is the area integral of the curl of A to a line integral over A where we use loop 2 as the path. We next write A using our integral expression. Recall we used this integral expression to calculate the vector potential of a spinning ball of charge, and to develop the magnetic multipole expansion. This also involves a loop integral but this time its loop 1 which creates the vector potential A. This loop 1 integral involves the 1/|r – r'| where r' is the "source" point (a position in loop 1) and the "observation point" in this case a position in loop 2. Chaining these expressions together we get a double line integral expression for the flux which easily converts to a double integral expression for the mutual inductance M_12. The Neumann Eq. gives a nice "cook-book" way of computing the mutual inductance on a computer. All you need to do is parameterize both loops and a computer can do the double sums in a flash. The homework will convince you that it isn't particularly easy to do analytically. The real interest in the Neumann formula is based on the following observation. We see that the Neumann Eq. says that the roles of 1 and 2 can be reversed in the integral and it will make no difference. This is because we can always reverse the order of integration and |r_1 - r_2| = |r_2 – r_1|. We thus conclude M_12 = M_21. A very quick demonstration that M_12 = M_21 is provided by two common core solenoids as you would get in a transformer. Both are "long" solenoids such that L is much larger than the transverse dimensions "a", and "b". We calculate first calculate the field of due to the inner solenoid using the solenoid formula. We then integrate the flux through the outer solenoid which is simple since the inner long solenoid field is uniform up to radius "a" and then cuts off so the contributing flux area is pi a^2. We then redo the calculation reversing the role of solenoid "a" and solenoid "b". Now the big solenoid creates the field, and we calculate the flux through the little solenoid. We get the same answer for both calculations and conclude M_ab = M_ba as expected from the Neumann formula. Although M_12 = M_21 looks fairly reasonable given the two solenoid example, Griffiths points out how implausible this really is. For example, imagine computing the mutual inductance between a single circular loop and a long solenoid. The calculation is very simple if the solenoid creates the field, and we compute the flux through the single loop, but imagine reversing the roles. The field of a single loop is complicated enough on axis as we found, but it is very complicated off of the symmetry axis. It also dies off very quickly away from the center of the loop. To compute the mutual inductance we would have to integrate the field from the single loop over each point contained in each of the N_b loops of the solenoid. I've given a sketch of the field lines from the small loop and how complicated this calculation might be for just four loops. But we don't need to do this very challenging calculation since M is a snap if solenoid creates the field, and we compute the flux through the smaller loop. The Neumann formula is very easy to derive, very elegant in appearance and produces some implausible – but correct– conclusions.

Self inductance and magnetic energy





$$\vec{B} = \mu_0 \frac{NI}{\ell}\hat{\eta} \; ; \; \Phi = N\pi a^2 \hat{\eta}\cdot\vec{B}$$

$$\Phi = \pi a^2 \mu_0 \frac{N^2 I}{\ell} \; ; \; L = \frac{\Phi}{I} = \frac{\mu_0 \pi a^2 N^2}{\ell}$$

$$\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} \; ;$$

emf opposes $\frac{dI}{dt} \leftarrow$ Lenz's Law

$$V - L\frac{dI}{dt} = 0 \rightarrow V = L\frac{dI}{dt}$$

$$\text{power } = VI = LI\frac{dI}{dt}$$

$$U = \int_0^\infty \text{power } dt = \int_0^\infty LI\frac{dI}{dt}\,dt$$

$$U = \int_0^I LI'\,dI' = \frac{LI^2}{2} = \frac{\mu_0 \pi a^2 N^2 I^2}{\ell}$$

$$U = \frac{1}{2\mu_0}\left(\frac{\mu_0 NI}{\ell}\right)^2 \pi a^2 \ell = \frac{B^2}{2\mu_0} \times \text{volume}$$

Recall $U_e = \frac{\varepsilon_0 E^2}{2} \times \text{volume}$

10

You are probably more familiar with self-inductance which is the flux through a loop due to the B-field created by the loop itself when carrying a current. Again, a circular solenoid gives a super simple example. The B- field is the familiar mu_0 I N/L and is uniform. The flux is just the number of loops times the area of each loop and the self inductance is this flux divided by the current through the loop. If the current changes, the flux changes, and an emf develops in the direction of opposing the changes in the current in a way consistent with Lenz's Law. As an important example, consider steadily increasing the current through a solenoid using a variable voltage source from zero to a final current I. This will cause the magnetic field to rise from 0 to a final value B. As you recall from Physics 212, the sum of the voltage changes around a circuit is zero. This means the voltage must equal the inductance times dI/dt. The inductance supplies a voltage "drop" equal to dI/dt. The instantaneous power delivered by the voltage source is just V times I. The total energy supplied by the voltage source in setting up the B-field will be the time integral of the power. We will give ourselves plenty of time to set up the field. We can change the integral from dt to dI as a variable substitution. The integration limits will change from 0 to infinity in time to 0 to I in current. We change I to I' to avoid confusion between the integrand and the limits. This is a trivial integral and yields a total energy of L I^2/2. Since there is no heat dissipation since there are no resisters, this energy presumably is stored in the solenoid – we can think of it as being stored in the magnetic field. We now replace L by the self inductance formula for a solenoid we just derived. And we notice that our expression can be written in terms of the square of the B-field and the volume of the solenoid. The formula looks highly analogous to the energy storage in an E-field except epsilon_0 gets converted to 1/mu_0. This always seems to happen when one goes from electricity to magnetism!

More General Magnetic Energy

$$U = \frac{LI^2}{2} = \frac{I}{2}\Phi = \frac{I}{2}\oint \vec{A}\bullet d\vec{\ell}$$

A reasonable generalization

$$\oint (\ ) I d\vec{\ell} \rightarrow \int_V (\ ) \vec{J} d\tau$$

$$U = \frac{1}{2}\int_V \vec{A}\bullet\vec{J} d\tau = \frac{1}{2\mu_0}\int_V \vec{A}\bullet\vec{\nabla}\times\vec{B} d\tau$$

We now use an IBP trick

$$\vec{\nabla}\bullet\left(\vec{A}\times\vec{B}\right) = \vec{B}\bullet\vec{\nabla}\times\vec{A} - \vec{A}\bullet\vec{\nabla}\times\vec{B}$$

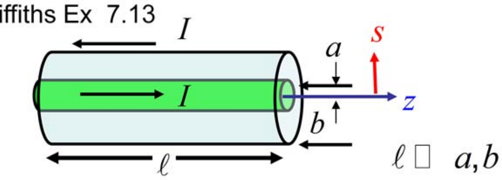
$$\rightarrow \vec{A}\bullet\vec{\nabla}\times\vec{B} = \vec{B}\bullet\vec{B} - \vec{\nabla}\bullet\left(\vec{A}\times\vec{B}\right)$$

$$U = \frac{1}{2\mu_0}\int_V \vec{B}\bullet\vec{B} d\tau - \frac{1}{2\mu_0}\int_V \vec{\nabla}\bullet\left(\vec{A}\times\vec{B}\right) d\tau$$

$$U = \frac{1}{2\mu_0}\int_V \vec{B}\bullet\vec{B} d\tau - \frac{1}{2\mu_0}\int_S \left(\vec{A}\times\vec{B}\right)\bullet d\vec{a}$$

$$\boxed{U = \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}\bullet\vec{B}\, d\tau}$$

Griffiths Ex 7.13



Find energy stored in coaxial cable

$$U = \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}\bullet\vec{B}\, d\tau$$

$$d\tau = 2\pi s\ell\, ds \ ;\ 2\pi s B = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$U = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 \int_a^b \frac{2\pi s\ell\, ds}{s^2}$$

$$\rightarrow U = \frac{\mu_0 I^2 \ell}{4\pi}\ln\left(\frac{b}{a}\right) = \frac{LI^2}{2}$$

$$\rightarrow L = \frac{\mu_0 \ell}{2\pi}\ln\left(\frac{b}{a}\right)$$ A freebee!

11

We can generalize the argument for the magnetic field energy somewhat (non-constant fields and arbitrary volumes) although we still follow essentially the same script. We start with our LI^2/2 expression. Our next step is to recycle the (by now) familiar trick of writing the flux as the line integral over the vector potential. It is always best in a derivation to try to go to a volume integral over J rather than a line integral over I dL. If we convert to J we can use the differential form of Ampere's law which is much more elegant than the integral form. The obvious replacement is I dL turns into J d Volume. This is the same replacement we used in discussing the Biot-Savart law. We replace J with the curl of B over mu_0 to get an expression entirely in terms of "field-type" quantities such as the vector potential (whose curl is the B-field) and the B-field itself. If only we could move the curl from the B-field to the curl of the vector potential! We would then have B dotted into itself integrated over the volume which would agree with our solenoid expression. But, of course, moving a derivative from one term in a product to another is why integration by parts was invented! Indeed the correct product rule involving curls is in the first chapter of Griffiths. The second term can be converted into an area integral since it is a divergence. If we thus have an expression that looks very similar to the electrostatic case (Griffiths Eq. 2.44). If we integrate over all space, the surface piece will vanish since B will vanish at infinity and we end up with an energy density of B^2/(2 mu_0) which looks similar to the electrostatic energy density of (epsilon_0/2) E^2 -- and again epsilon_0 converts to 1/mu_0. As an example of magnetic energy we compute the energy stored in a coaxial cable which consists of an inner conductor and an outer conductor. Here the current flows along the inner conductor and returns along the outer conductor. We can easily calculate the B-field using Ampere's law. Outside of the cable , there will be no B-field since the net enclosed current about a circular path is zero. This means we perform the integral in the region 0 < Z < L and s from "a" to "b". We get a stored energy U which is proportional to the logarithm of b/a which frequently happens in cylinder problems. Since U = LI^2/2, we can use this expression to compute the self-inductance L to get two useful results for the price of one.

Why they call them Maxwell's Eq.

The Pre-Maxwell Eq.

$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho$ (Gauss's Law)

$\vec{\nabla} \bullet \vec{B} = 0$ (no monopoles)

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's Law)

But these are inconsistent with the Continuity Eq.

$\vec{\nabla} \bullet \vec{J} + \frac{\partial \rho}{\partial t} = 0$ (Q Conservation)

The problem is with Ampere

$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \vec{\nabla} \bullet \vec{J} = 0$

(since curl of divergence is 0)

What happened to $\frac{\partial \rho}{\partial t}$??

Can we restore $\frac{\partial \rho}{\partial t}$ using Gauss?

$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho \rightarrow \varepsilon_0 \vec{\nabla} \bullet \frac{\partial \vec{E}}{\partial t} = \frac{\partial \rho}{\partial t}$

$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \left[\vec{\nabla} \bullet \vec{J} + \varepsilon_0 \vec{\nabla} \bullet \frac{\partial \vec{E}}{\partial t} \right] = 0$

$\rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}} \leftarrow$ Maxwell

A symmetry w/ Faraday's Law!
and much much more!

Maxwell Equations 1861

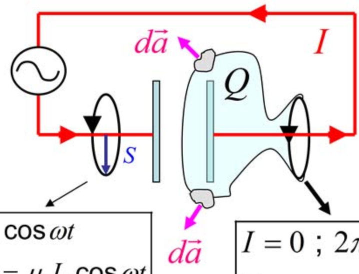
$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho \quad ; \quad \vec{\nabla} \bullet \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \; ; \; \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

12

Here is a summary of our main “field” formulas in differential form prior to Maxwell. They are actually internally inconsistent with the conservation of charge in the form of the continuity equation. The problem is that Ampere’s law is incomplete as written. This is very easy to show. If we take the divergence of both sides of Ampere’s law, on the left side we have the divergence of a curl which is always zero and on the right we have the divergence of the current density. The continuity equation says that the only way the divergence of J can be zero is if there is no time dependence to the charge density. This was indeed the case for electrostatics and magnetostatics but isn’t true in electrodynamics. The simplest thing to do is force consistency with the continuity equation by somehow adding an extra term to Ampere’s law that brings in partial rho / partial t when one takes the divergence of the curl of B. The obvious way to do this is to add the time derivative of E since the divergence of E is rho. We are then driven to adding a partial E/partial t term to Ampere’s laws. This has the additional virtue of restoring more symmetry between the electric and magnetic fields in our field equations. Just like the curl of E involves the rate of change of B, the curl of B involves the rate of change of E post-Maxwell. Griffiths says this wasn’t Maxwell’s main motivation to adding the new term to Ampere’s law but it is clearly very compelling and easy to motivate. It opens up a dramatic new possibility of creating an electromagnetic system that can exist in a vacuum with no charges or currents. A changing E field can now create a changing B-field which can in turn create a changing E-field – sort of a self generating dynamo. The Maxwell term supplies the missing link which makes this dynamo possible and allows electric and magnetic fields to exist in a charge-less vacuum. We are of course describing light and radio waves.

Maxwell displacement current



$$\int_S \vec{\nabla}\times\vec{B}\bullet d\vec{a} = \mu_0\int_S \vec{J}\bullet d\vec{a} + \mu_0\int_S \varepsilon_0\frac{\partial\vec{E}}{\partial t}\bullet d\vec{a}$$

$$\oint \vec{B}\bullet d\ell = \mu_0 I + \mu_0 I_D \text{ where } I_D = \varepsilon_0\frac{\partial\Phi_e}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\vec{J}_D \text{ where } \vec{J}_D = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$I = I_0\cos\omega t$

$2\pi sB = \mu_0 I_0\cos\omega t$

$\vec{B} = \frac{\mu_0 I_0\cos\omega t}{2\pi s}\hat{\phi}$

$I = 0\ ;\ 2\pi sB = \mu_0 0$

$\vec{B} = 0???$

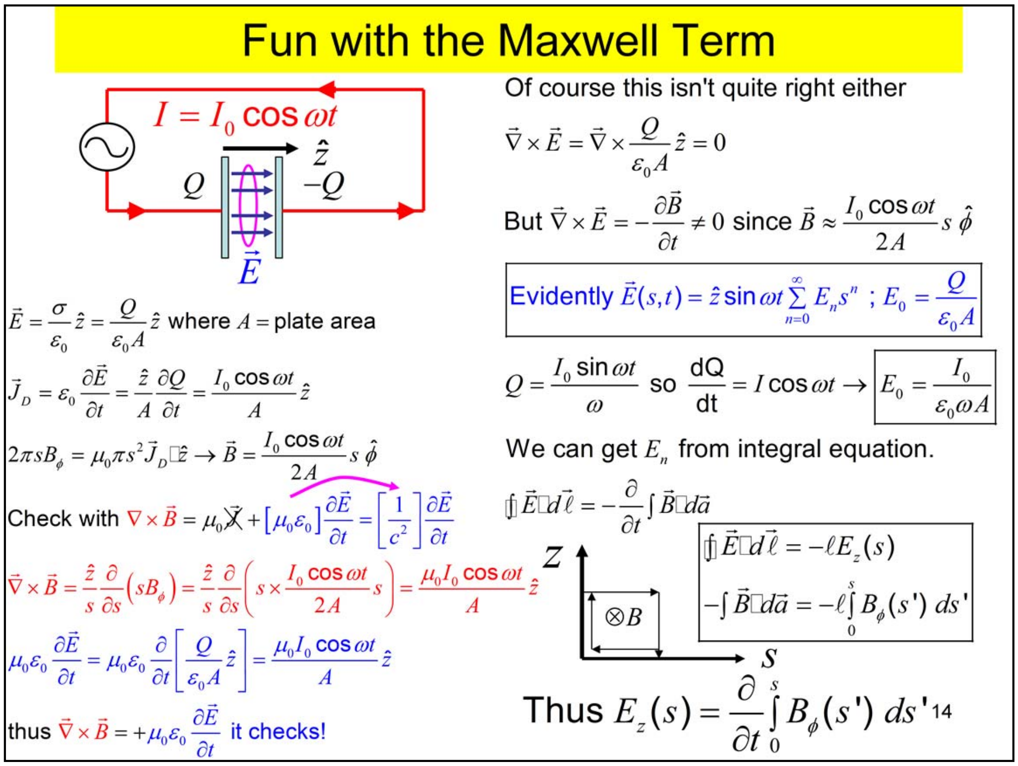
$I_D = \varepsilon_0\frac{\partial\Phi_e}{\partial t} \quad \to \varepsilon_0\Phi_e = Q$

$I_D = \frac{\partial Q}{\partial t} = I = I_0\cos\omega t$

$\to \vec{B} = \frac{\mu_0 I_0\cos\omega t}{2\pi s}\hat{\phi}$

13

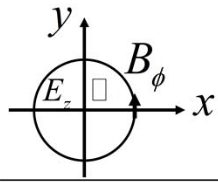
Here is another, common way of viewing the internal inconsistency of Ampere's law. If we make the jump from electrostatics to electrodynamics, we can pass an oscillating current though a capacitor by driving the capacitor with an oscillating voltage. We can compute the B-field from this current (in the infinite wire limit) using integral form of Ampere's law which says the line integral of the B-field equals the integral of J passing through a bounding surface. In the case of the Amperian construction on the left this J integral is just the current I and we get the usual form for B for a long wire. The obvious "bounding" surface is just a circular disk. On the other hand, the Stokes' theorem never says that one must use the obvious bounding surface, **any** bounding surface will work. We illustrate an unconventional choice for a bounding surface on the right, we choose a satchel shaped surface which encompasses the capacitor plate and then flares up to bound the loop. We show some area vectors which must point normal to the surface in a way consistent with RHR (of the B-loop) as you show in HW. There is no current density that flows through the satchel volume but we can't conclude there is no B-field since we get a perfectly reasonable B-field using another "bounding" surface. The Maxwell term rescues us from this contradiction when we write the Ampere-Maxwell Eq in integral form by making a surface integral of the curl of B and then using the Stokes' Thm. The Maxwell term adds a new "current" called the displacement current which is proportional to the rate of change of electric flux through our Amperian loop. This language is essentially equivalent to writing the curl of B in terms of a displacement current density which is essentially the time derivative of the D-field. The surface integral of J_D is the displacement current. We now can get back to the satchel surface calculation of the B-field. The total current relevant here is the displacement current since there is no physical current. If we make the satchel neck very small the satchel essentially encloses the charge Q on the right capacitor plate and thus epsilon_0 times the satchel flux is Q according to Gauss's Law and the displacement current is the rate of change of Q which "happens" to be the physical current. We thus get the same prediction for the B-field on the right side as we did on the left side.



In the previous example, we computed the B-field around one of the wires, we turn next to the field within the capacitor plates and work in the limit of an “infinite,small-gap” capacitor. As before we use a cos omega t current. In this case we know the E-field is in the z-direction with a strength of sigma/epsilon_0. The displacement current density is essentially the derivative of this E-field which is proportional to the derivative of Q or the physical current. We now compute B using a hopefully familiar Maxwell-Ampere loop argument which sets the line integral of B to the integral of the uniform displacement current. We thus have B-field in the phi-hat direction which is proportional to the cylindrical coordinate s. We can check this form for B by using the Maxwell-Ampere law in differential form. We will anticipate Physics 436 (where we obtain the wave equation) and write epsilon_0 mu_0 as 1/c^2. We simply take the curl of our B-expression in cylindrical coordinates. Indeed the curl of our B-field is proportional to the rate of change of E. Now of course in reality none of the forgoing calculations is technically correct since our initial expression for the E-field isn’t correct. We started with a constant E-field with zero curl. But Faraday’s law says the curl of E is the rate of change of B and we indeed have a B-field in the gap. To describe the inevitable non-uniform field, we develop a Taylor expansion in s times a sin (omega t) time dependence. This means all terms have the time dependence of the leading (E_0) term which is proportional to Q or the integral of the cos omega t current. We get rest of the terms by developing a recursion relation which we get by setting up a “consistency” integral equation based on finding E from Faraday’s law involving the integral over B, and using the Ampere-Maxwell law to write B as an integral over E. The plan is that E_0 term is due to the charges on the plate while the recursion relation describes the higher order terms due to Faraday’s law. We illustrate the Faraday law part. We use a E-loop in the z-s plane. We place one leg along the z-axis where we know (from our 1st order B-expression) that there is no B field and thus no B-field contribution. Recall the electrostatic contribution is confined to the E_0 term.

Integral equation and series solution

$$E_z(s) = \frac{\partial}{\partial t}\int_0^s B_\phi(s')\,ds'$$



$$\oint \vec{B}\cdot d\vec{\ell} = \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{a}$$

$$2\pi s' B_\phi(s') = \frac{1}{c^2}\frac{\partial}{\partial t}\int_0^{s'} 2\pi s''\,ds''E_z$$

$$B_\phi(s') = \frac{1}{s'\,c^2}\frac{\partial}{\partial t}\int_0^{s'} s''\,ds''E_z$$

Thus

$$E_z(s) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_0^{s}\frac{ds'}{s'}\int_0^{s'} s''\,ds''E_z$$

Inserting $E_z(s) = \sin\omega t\sum_{n=0}^{\infty} E_n s^n$

$$\sum_{m=0}^{\infty} E_m s^m = \frac{-\omega^2}{c^2}\sum_{n=0}^{\infty}\int_0^{s}\frac{ds'}{s'}\int_0^{s'} s''\,ds''(s'')^n E_n$$

$$\int_0^{s'} s''\,ds''(s'')^n E_n = \frac{(s')^{n+2}E_n}{n+2}$$

$$\int_0^{s}\frac{ds'}{s'}\frac{(s')^{n+2}E_n}{n+2} = \frac{E_n s^{n+2}}{(n+2)^2}$$

$$\to \sum_{m=0}^{\infty} E_m s^m = \frac{-\omega^2}{c^2}\sum_{n=0}^{\infty}\frac{E_n s^{n+2}}{(n+2)^2}$$

$$m = n+2$$

Thus $E_m s^m = \frac{-\omega^2}{c^2}\frac{E_n s^{n+2}}{(n+2)^2}$

or $E_{n+2} = -\frac{1}{(n+2)^2}\left(\frac{\omega}{c}\right)^2 E_n$

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We next find the B-field using the Ampere-Maxwell law. We again write this in terms of 1/c^2 rather than epsilon_0 mu_0. This is just a slight generalization of our previous B expression where the loop is circle in the x-y plane and we write a 2 pi s" ds" area integral over E_z. We use s" as an integration variable in anticipation that it will be inserted into our integral for E. We are left a consistency condition which says E can be written as a double integral over E. We can use this to relate the E_n coefficients in our series solution. We sum over m on the LHS and n over the RHS since we will find the powers are not the same. The double derivative with respect to time brings in a factor of – omega^2 and allows us to cancel the sin(omega t) factor. We now are left with doing the very simple double integral of the power law to finally get an expression relating the m and n series. In order to match the s powers , m = n+2 giving us a recursion relation which relates the E_(n+2) term to the E_n term.

The capacitor standing wave

$$E_z = \sin\omega t\left(E_0 + E_1 s^1 + \boxed{E_2 s^2 + E_3 s^3 \cdots}\right)$$

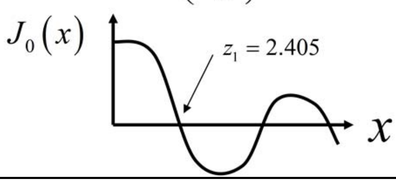
E_0 & E_1 fit BC for 2nd order DE but

$E_{n>1}$ follow from R.R. $\boxed{E_{n+2} = -\frac{1}{(n+2)^2}\left(\frac{\omega}{c}\right)^2 E_n}$

Here $E_0 = \frac{I_0}{\varepsilon_0 \omega A}$ and $E_1 = 0$; $E_2 = -\frac{E_0}{2^2}\left(\frac{\omega}{c}\right)^2$; $E_4 = -\frac{E_2}{4^2}\left(\frac{\omega}{c}\right)^2$

$$\vec{E} = \hat{z}E_0 \sin\omega t\left[1 - \frac{1}{2^2}\left(\frac{\omega\, s}{c}\right)^2 + \frac{1}{2^2\left(4^2\right)}\left(\frac{\omega\, s}{c}\right)^4 - \frac{1}{2^2\left(4^2\right)\left(6^2\right)}\left(\frac{\omega\, s}{c}\right)^6 \cdots\right]$$

$$\vec{E} = \hat{z}E_0 J_0\left(\frac{\omega\, s}{c}\right)\sin\omega t$$



J_0 is a Bessel Function and $\vec{E} = \hat{z}E_0 J_0\left(\frac{\omega\, s}{c}\right)\sin\omega t$ is a standing wave.

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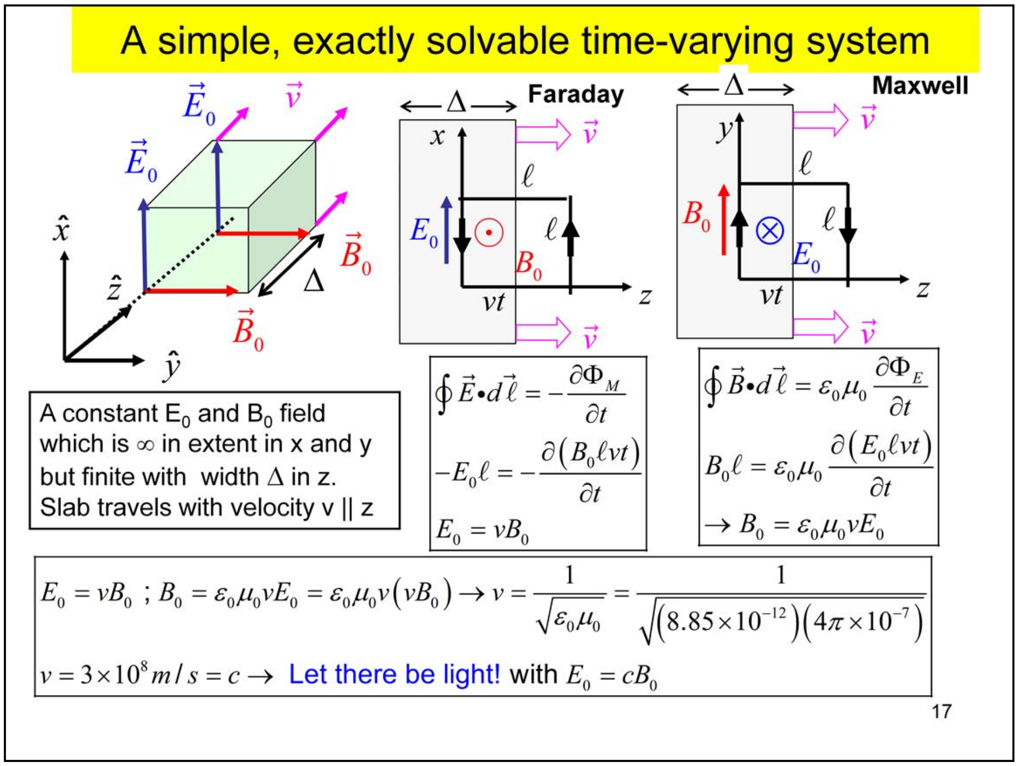
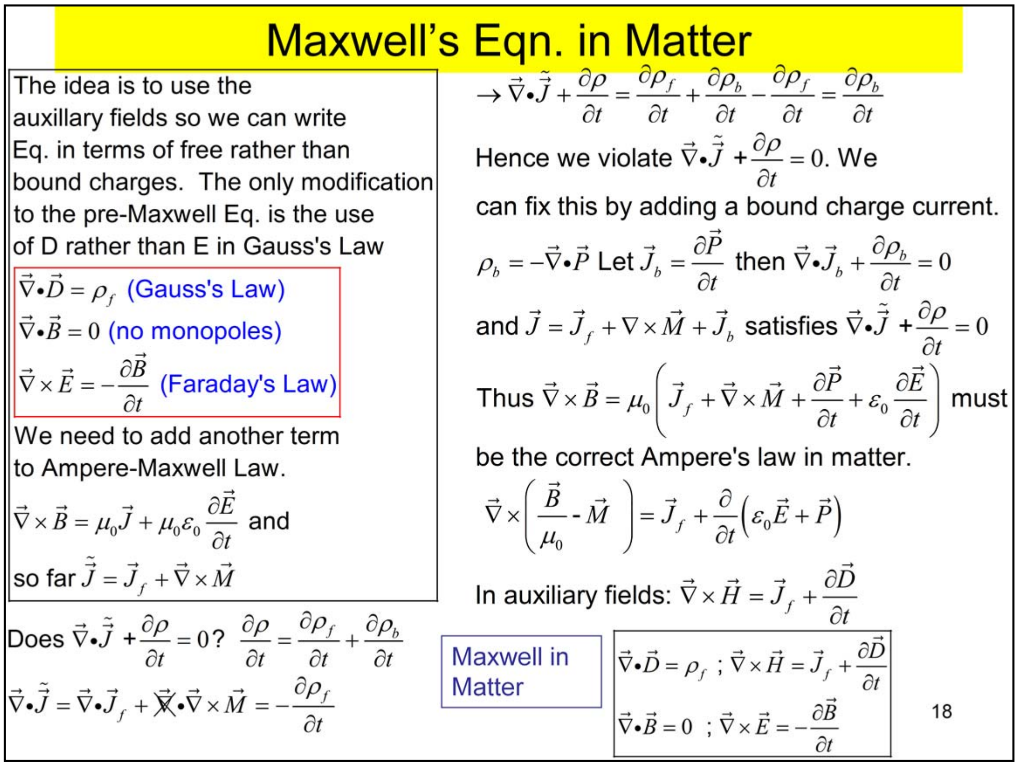
Writing out the series expression, we note that the recursion relation allows us to find E_2 from E_0 and E_3 from E_1 but we have no way of finding E_1 from E_0 since we only relate E_(n+2) to E_n from our integral expression. Essentially E_0 and E_1 are the two parameters required to match the boundary conditions of a second order differential equation. As we will learn in Physics 436, the E-field satisfies a second order differential equation called the wave equation. A similar thing happens in the second order differential equation describing one dimensional motion of a particle in force field. One needs to specify the initial position and initial velocity. Here we get E_0 from the electrostatic contribution, and E_1 is zero. We next use the recursion relation and the form of our power series to write the E-field as a polynomial in (s omega/c) times sin omega t. This series is the Taylor expansion of a Bessel function which is a quasi-periodic function.

If the oscillating frequencies aren't too large, our zero order result for E which is uniform in s should be accurate enough. The palliative phrase is our results are ok in the quasi-static limit. In Physics 436 we show, an oscillating field creates a radio wave with a wavelength lambda where 2pi /lambda = omega/c and c is the speed of light. Here we have a standing wave where the field is the product of a space times a time piece. I think that as long as lambda is much larger than the relevant dimension of our components (such as the capacitor gap) we should be comfortably in the quasi-static limit where E is given by the 1st term.



It looks like the interplay of Faraday's Law and the Maxwell term inevitably lead to an electromagnetic system with a complicated series solution. The oscillating E-field creates an oscillating B-field which in turn creates a higher order correction to the E-field ad nauseum resulting in an infinite series in omega. We generally have the choice of summing this infinite series or working in the "quasi-static approximation" where only the lowest order omega terms are kept. This is indeed true for most cases where the electromagnetic fields are created by an oscillating current or charge source. However there is a historically important example which allows electric and magnetic fields to exist in a vacuum with no charge or currents – the changing E field creates a changing B field and vice versa. This is the physics behind the electromagnetic wave which can often be solved by very simple mathematics as we will show in Physics 436. Here is a preview of a very simple system which consists of a slab of constant electric field along the x axis and a slab of constant magnetic field along the y axis which moves with a constant velocity v along the z axis. We are able to get changing electric and magnetic fluxes by giving the slab a finite thickness of Delta in z and a z-velocity. (Note the slab can only have thickness in z since a infinite extent in the x and y field directions implies a non-zero divergence of E or B which requires a non-zero charge or monopoles density.) We can relate E to B by applying Faraday's law to a square L by L loop in the x-z plane. To get a useful relation we have to imagine that the field slab partially penetrates the Faraday loop as shown. We use a counter clockwise loop in accordance with our right-hand rule convention. The total line integral is –E0 L since only the left side has a non-zero E-field. The total rate of change of magnetic flux is B0 L v hence Faraday's law tells us E0 = v B0. We apply the displacement current loop in the y-z plane using a clockwise loop according to the RHR. The displacement current loop gives us another simple electric-magnetic relation: B0 = epsilon_0 mu_0 v E0. We can get a consistent electromagnetic field only if the slab moves with a velocity of v= 1/sqrt(epsilon_0 mu_0) = 3e8 m/s with respect to the Faraday and displacement loops. We have "derived" the velocity of electromagnetic waves using very simple and straightforward applications of Faraday and Maxwell loops. Our derivation is very similar to the EMF calculation of a loop moving through the magnetic field– only here the fields move with respect to the loop. Our result is consistent with the experimental value speed of light measured by Fizeau shortly before Maxwell's work – leading to the striking conclusion that light is a "source-less" electromagnetic wave. One could call this the crowning achievement of classical physics. It also has the seeds of relativity. We found that the slab velocity is c with respect to any stationary Faraday-Maxwell loop. If we think of the loop as defining the reference frame, the electromagnetic wave velocity is independent of reference frame unlike the velocity of all other classical objects. We will show in Physics 436 that the only way this can happen is if the length of time intervals are reference frame dependent which is the foundation of the theory of relativity.



Having completed one form of Maxwell Eq. involving total charge and total current, we will generate an alternative form using the auxiliary fields D and H which references just free (controllable) charges and currents. The vanishing of the B divergence, and Faraday's law makes no reference to charge or currents and are unaffected. We already wrote Gauss's Law in terms of free charges using the D-field and of course there are no monopole charges so that job is done. This leaves Ampere's law to reconsider. This writes the curl of B in terms of the current density and the displacement current density (essentially the rate of change of E). Thus far we have considered the current density of free charges as well as bound currents due to atomic orbitals which is given by the curl of the magnetization. We will call the curl of M current: the magnetization current. Interestingly enough there must be an additional current since we can show the divergence of these two currents does not account for the rate of change of the total charge density and thus violates the continuity equation. To demonstrate this lets assume that J-tilde is the sum of the free and "magnetization" current. The divergence of J-tilde is the divergence of J_f plus the divergence of the magnetization current. But the later is the divergence of a curl and disappears and divergence of J_f is the negative of rate of change of the free charge density. If we combine the divergence of J-tilde with the rate of change of the charge density we won't get zero as the continuity equation demands but will be left with the rate of change of the bound charge density. This bound charge density is due to polarization of the matter due to the electrical field and thus has nothing to do with the magnetization current. Recall the bound charge density is the negative of the divergence of the polarization. We thus can cure the problem by adding a new bound current equal to the rate of change of the polarization vector. This form with the **four** currents (free, magnetization, bound, and displacement), can be re-arranged to give an elegant expression in terms of H and D. It also clears up a linguistic mystery. We now know why the Maxwell term is called the displacement current.