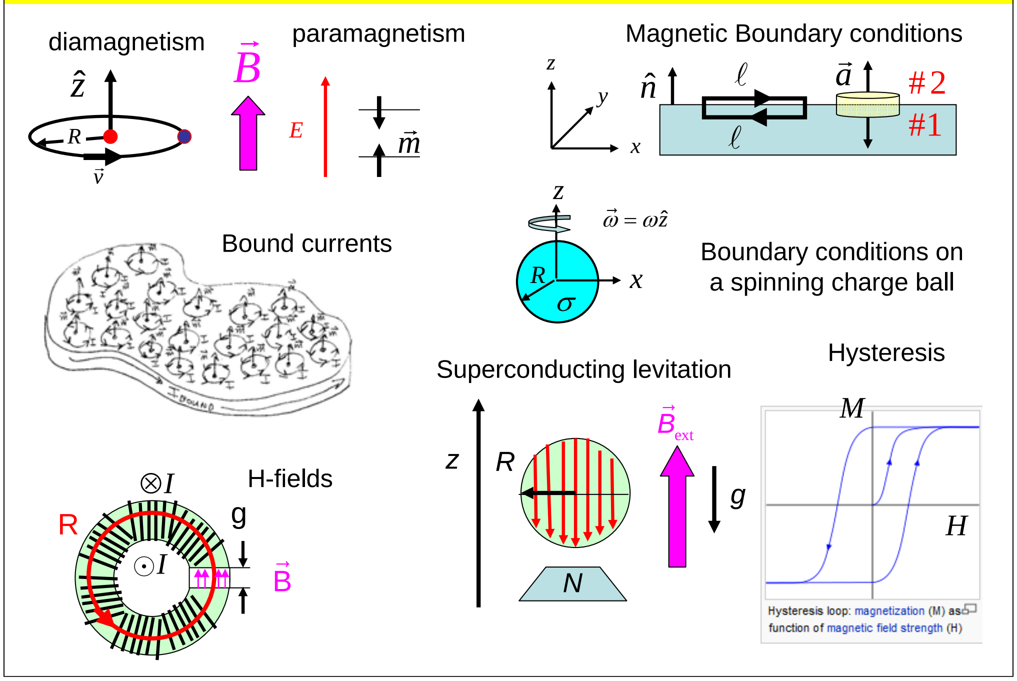
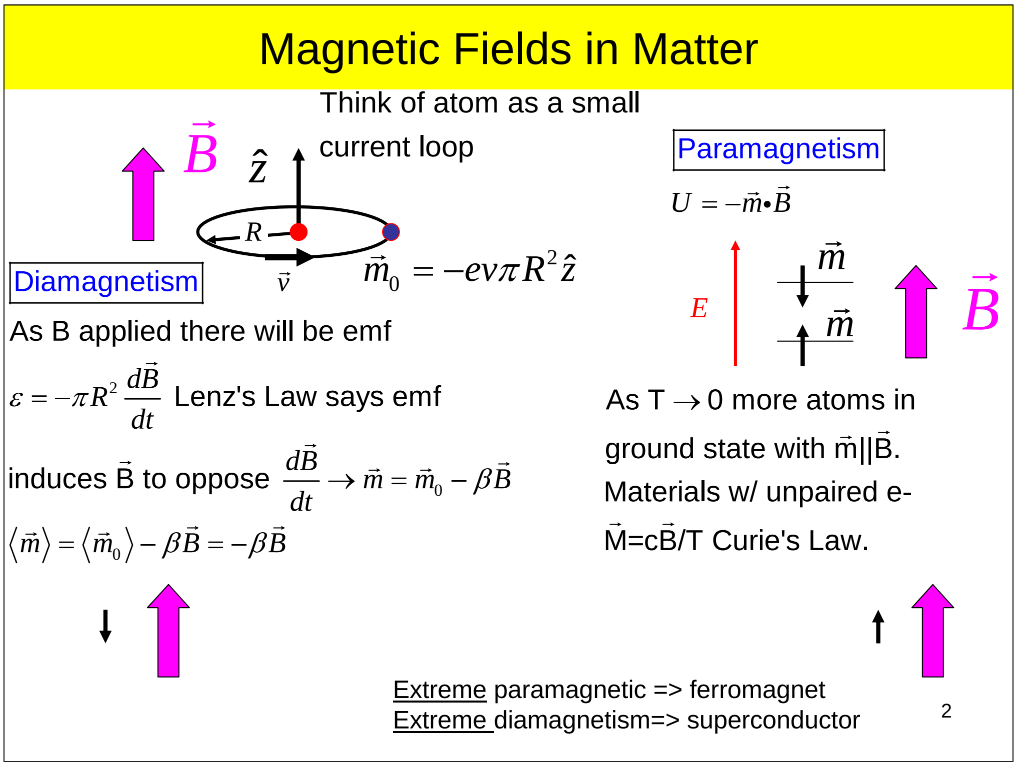
Magnetic Fields in Matter



There are several ways that materials can interact with magnetic fields. The first of this is called diamagnetism where an applied magnetic field induces a magnetic moment in the material that opposes the applied field. We can think of this as turning on an applied B-field induces an EMF in the atomic orbitals that opposes the B-field in accordance with Lenz's law which you learned about in Physics 212. Hence all materials exhibit some diamagnetisation. In materials with an odd number of electrons, there is a competing effect called paramagnetism which often overwhelms the diamagnetization. Essentially the magnetic moment of the unpaired electron lines up with the applied B-field in order to minimize its energy. In much the same way as an electric field creates bound charges on the surface of a dielectric, a magnetic field creates bound currents on the surface of a magnetic material. Just like the D-field has a Gauss's law that only depends on free charges, the H-field has an Ampere's law that is only sensitive to free currents. We next discuss the boundary conditions that apply on the surface of a magnetic material, and show that our spinning ball example from last time obey's these boundary conditions. We conclude by discussing superconductors which can be thought of as an extreme diamagnet and ferromagnets which can be thought of as an "extreme" paramagnet.



The interaction of magnetic fields and materials is essentially an atomic phenomena and can only really be described using quantum mechanics. There are basically two competing effects called diamagnetism and paramagnetism. In diamagnetism, the atomic current loop reacts to the applied B field in a way to oppose the applied field by creating an induced magnetic moment that opposes the field. There are several classical ways of viewing this. Griffiths gives a classical argument for diamagnetism based on having the Lorentz force increase the centripetal force and the electron speed. Of course this is providing work on the electron and magnetic fields do no work. The work must be provided by a changing magnetic field, creating an EMF according to Faraday's Law. As you recall from Physics 212, this EMF is in the direction to oppose the change in flux through the orbit and hence it is naturally that the induced field opposes the applied field. Of course the orientation of the atomic dipoles is random but the induced change in the moment always opposes the applied B and hence the average m opposes B.

Paramagnetism is a competing effect that puts the induced dipole moment parallel to B. The recall lowest potential energy state (U) has m parallel to B. Classically m || B is the stable equilibrium direction for the dipole orientation and any torque will try to get the m-vector parallel to B. We can also discuss the problem in a quantum framework. At low temperatures the atoms will preferentially populate the ground state. As the temperature increases, thermal fluctuations will randomize the states and the paramagnetic preference will wash out, thus M goes as 1/ temperature according to the Curie Law. Paramagnetism tends to dominate when materials have an unpaired electron (odd Z). The paired electrons have canceling spins. Both diamagnetism and paramagnetism tend to be fairly weak effects. But there are two extreme examples called ferromagnetism and superconductivity that we discuss later.

Bound currents

The induced $\vec{\mathrm{m}}$ parameterized as $\vec{\mathrm{M}}=\dfrac{\vec{\mathrm{m}}}{\mathrm{vol}}$

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}\rightarrow\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int_V\frac{\vec{\mathrm{M}}(\vec{\mathrm{r}}')\times\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau'$$

where $\vec{\mathscr{r}}=\vec{r}-\vec{r}\,'$

$$\vec{\nabla}'\frac{1}{\mathscr{r}}=\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\rightarrow\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int_V\vec{\mathrm{M}}(\vec{\mathrm{r}}')\times\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

We next exploit a "curl" version of the integration by parts trick used in Polarization Chp

$$V(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int\vec{P}(\vec{r}\,')\bullet\nabla'\left(\frac{1}{\mathscr{r}}\right)\mathrm{d}\tau'$$

$$V(\vec{r})=\frac{1}{4\pi\varepsilon_0}\left[-\int_V\frac{\vec{\nabla}'\bullet\vec{P}}{\mathscr{r}}\,d\tau'+\int_S\frac{\vec{P}(\vec{r})\bullet d\vec{a}}{\mathscr{r}}\right]$$

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int_V\frac{\vec{\nabla}'\times\vec{\mathrm{M}}(\vec{\mathrm{r}}')\,d\tau'}{\mathscr{r}}+\frac{\mu_0}{4\pi}\int_V\frac{\vec{\mathrm{M}}(\vec{\mathrm{r}}')\times d\vec{a}\,'}{\mathscr{r}}$$

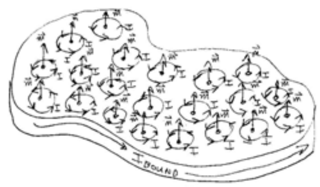
Comparing these forms to

$$\vec{A}(r)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')}{\mathscr{r}}d\tau'+\frac{\mu_0}{4\pi}\int_S\frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}da'$$

We identify $\boxed{\vec{J}_b(r')=\vec{\nabla}'\times\vec{\mathrm{M}}(\vec{\mathrm{r}}')}$

and $\boxed{\vec{K}_b(\vec{r}\,')=\vec{\mathrm{M}}(\vec{\mathrm{r}}')\times\hat{n}}$

We thus have "bound" volume and surface currents



Example of bound surface current. The inner loops cancel but the outer loops don’t leaving a net surface current.

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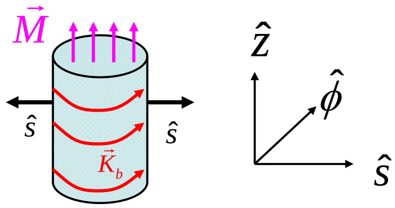
As is the case for the response of materials to electric fields, the response to magnetic fields is parameterized by an induced dipole moment per unit volume. In the magnetic case this is a magnetic dipole moment which is essentially the electron orbital current times the orbit area. The magnetic case is basically a recycled version of the electric case. The electric case is based on the voltage contribution of an electric dipole moment. The magnetic case uses the vector potential for a magnetic dipole which we derived in the magnetostatics chapter. We change the magnetic dipole to a magnetic dipole density (or magnetization) and then into a bound current J_b and bound surface current K_b using the similar tricks as those used for the electric dipole polarization. We begin by writing the r-hat/r^2 that appears in the dipole vector potential as the gradient of 1/r. We use integration by parts to transfer the del operator from 1/r to M (the magnetic dipole density) plus a surface term. The resultant integrals look the same as the vector potential for a current density J and a surface current K. For the magnetic case these are related to the cross product of del and the magnetization for J (eg the curl) and the cross product of the magnetization density (M) and the area normal for K. For the electric case we used dot products with the polarization (P). Griffiths emphasizes that although we obtained the current densities using some slick mathematical tricks – they represent real “bound” surface (K) and volume currents (J). This sketch shows how the surface current develops from a collection of atomic orbitals. Adjacent orbitals in the interior of the material produce cancelling currents. But orbitals on the material surface have no canceling currents and hence a bound surface current can exist. Many of the magnetic problems discussed here involve just the bound surface currents just like many of the electric problems just involve bound surface charge. But in principle both bound surface currents and volume currents can be present.

Uniformly Magnetized Cylinder

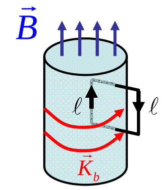
Uniformly magnetized cylinder



$$\vec{M} = M_0\hat{z} \rightarrow \nabla\times\vec{M} = 0 \rightarrow \vec{J}_b = 0$$

$$\vec{K}_b\left(\vec{r}\,'\right) = \vec{M}\times\hat{s} = M_0\ \hat{z}\times\hat{s} = M_0\hat{\phi}$$

Find $\vec{B}$ from $\vec{K}$ with Amperian Loop



$$\vec{K}_b = M_0\hat{\phi}$$

$$\oint\vec{B}\bullet d\vec{\ell} = B\ell = \mu_0 I_{enc} = \mu_0 K_b\ell$$

$$\rightarrow \vec{B} = \mu_0 K_b\hat{z} = \mu_0 M_0\hat{z} = \mu_0\vec{M}$$

Just like a solenoid!

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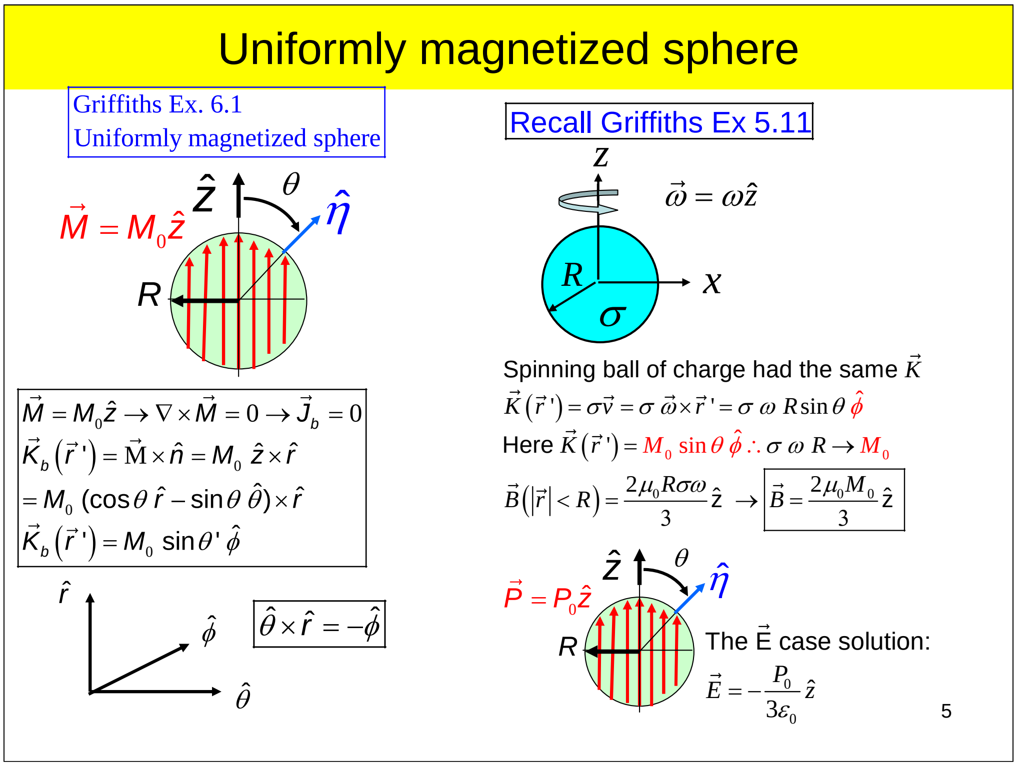
As a first (simple) example we consider a permanently magnetized (infinite) cylinder. We model this with a constant magnetic dipole density (or magnetization) M pointing in the z direction of a cylindrical coordinate system. The curl of a constant M is zero and hence there are no volume (J) currents. The surface normal is in the direction of the radial vector s-hat. Hence using the right hand rule you can verify that M cross s is in the phi direction. Hence the bound surface current circles the circumference of magnetized cylinder just like the currents of a solenoid. We can then solve for the magnetic field using the same Amperian loop approach used for the infinite solenoid. We use a loop perpendicular to K_b with an arbitrary length L_perp. One end of the Ampere loop extends into the solenoid and one extends outside of the solenoid in the field free region. Recall the total enclosed current for a surface current is equal to the perpendicular length which “cuts” K. We get the same simple result that B is mu_0 times the surface current which in this case means B is equal to mu_0 times the magnetization M inside the cylinder and 0 outside the cylinder. The situation for a finite length magnetized cylinder will be more complicated. There will be outside field lines which cycle from the north pole (top) to the south pole (bottom).



This example is based on Griffiths Ex. 5.11 -- the spinning charged metal sphere. Recall we solved this problem using a Laplace Eq. approach for the scalar magnetic potential V. We have a constant magnetization M aligned along the z axis of a spherical coordinate system. We first find the bound currents using the magnetization. Since we are using a constant M with zero curl, the bound J current is zero. We can find the bound surface current K by crossing the surface normal (r-hat) with M. We can do this by writing z-hat as a combination of the spherical unit vectors r-hat and theta-hat. Since r-hat cross r-hat is zero, we only get the theta-hat cross r-hat piece which is in the phi-hat (azimuthal) direction and is proportional to sin(theta). We get the same sin (theta) phi-hat form for the surface current in this case as we had for the spinning ball of charge. This allows us to steal the spinning ball results with the substitution (sigma omega R) => M. Hence we can purloin the expression for expression for the magnetic field inside the ball. As was the case for the magnetized cylinder, we find the magnetic field for the ball is proportional to mu_0 times the magnetization but there is an additional factor of 1/3 for the sphere which was not present for the cylinder. The uniform polarized sphere solution, requiring the full Laplace Eqn. machinery, we found an E= field proportional to the electric dipole moment per unit volume (P). For the electric case we get a denominator epsilon_0 factor rather than a numerator mu_0 factor. Interestingly enough for the electric case we have a minus 1/3 factor rather than the positive 2/3 factor that we have for the magnetic case.

The H-field

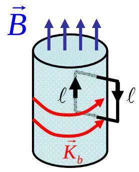
We recycle the trick used to define $\vec{D}$ which is only sensitive to free q:

$$\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}_f+\vec{J}_b\right)=\mu_0\left(\vec{J}_f+\vec{\nabla}\times\vec{M}\right)$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right)=\vec{J}_f\rightarrow\boxed{\vec{\nabla}\times\vec{H}=\vec{J}_f}$$

$$\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M}\rightarrow\vec{B}=\mu_0\vec{H}+\mu_0\vec{M}$$

$\vec{H}$ is only sensitive to $\vec{J}_f$

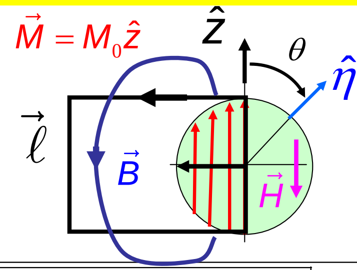


Revisit the magnetized cylinder

Recall there are no free currents.

$$\vec{\nabla}\times\vec{H}=\vec{J}_f \quad \oint\vec{H}\bullet d\vec{\ell}=H\ell=I_{enc}=K_f\ell=0$$

$$\therefore\vec{H}=0=\frac{\vec{B}}{\mu_0}-\vec{M}\rightarrow\vec{B}=\mu_0\vec{M}$$



Revisit the magnetized sphere

Recall there are no free currents.

Can we conclude $\vec{H}=0\rightarrow\vec{B}=\mu_0\vec{M}$?

Apparently not since $\vec{B}=\frac{2\mu_0\vec{M}}{3}$

What went wrong? $\vec{H}_<=\frac{1}{\mu_0}\frac{2\mu_0\vec{M}}{3}-\vec{M}=-\frac{\vec{M}}{3}$

Meaning $\int\vec{H}_<\bullet d\vec{\ell}<0$ evidently $\int\vec{H}_>\bullet d\vec{\ell}>0$

so $\oint\vec{H}\bullet d\vec{\ell}=0$. we can see $\int\vec{H}_>\bullet d\vec{\ell}>0$ from field lines.

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To further the strong analogy between magnetic and electrical response of materials, we invent a magnetic “auxillary” field called H in strong analogy with D-field (displacement field) which was the electric “auxillary” field. Like the D-field, H-field depends only on free currents. As a simplest example, we reconsider the magnetized, infinite cylinder. There are no free currents and Ampere’s Law says H = 0. This means M just cancels B and we immediately get the previous result. If we try to get B for the magnetized sphere in the same way (eg no free currents implying no H implying B cancels M) we run into problems. We get the wrong answer and miss the 2/3 factor. The problem is that no free currents only implies that the curl of H is zero which means line integral of H is zero. This is very different from the statement that H is zero everywhere. For the case of the sphere, there is an H-field outside the sphere which can contribute an H line integral that cancels the H line integral inside the sphere. For the sphere we don’t have enough symmetry to assume that a vanishing H line integral implies a vanishing H inside the sphere.

Definition of μ in linear media

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ For linear media $\vec{B} = \mu \vec{H}$

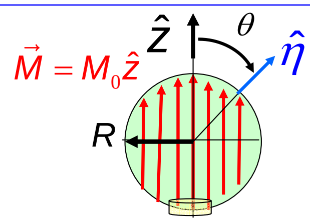
$\vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H} \to \vec{M} = \left(1 - \frac{\mu_0}{\mu}\right)\frac{\vec{B}}{\mu_0}$

$\vec{M} = \left(1 - \frac{1}{\kappa_m}\right)\frac{\vec{B}}{\mu_0}$ where $\kappa_m = \frac{\mu}{\mu_0}$

This is reciprocal of $\vec{E}$ case

$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$

$\vec{P} = \left(\frac{\varepsilon}{\varepsilon_0} - 1\right)\varepsilon_0 \vec{E} = (\kappa - 1)\varepsilon_0 \vec{E}$



$\vec{\nabla}\bullet\vec{B} = 0$. Is $\vec{\nabla}\bullet\vec{H} = 0$?

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \to \vec{\nabla}\bullet\vec{H} = -\vec{\nabla}\bullet\vec{M}$

Our sphere example has $\vec{\nabla}\bullet\vec{M} \neq 0$ on surface. $\int_V \vec{\nabla}\bullet\vec{M}\, d\tau = \int_S \vec{M}\bullet d\vec{a}$

For south pole $\int_S \vec{M}\bullet d\vec{a} = M_0 a$ from the pill box surface inside sphere.

If we mistakenly write $\vec{\nabla}\bullet\vec{H} = \rho_{mag}$

we would infer $\sigma_{mag} = \frac{-M_0 a}{a} = -M_0$

at south pole and $\sigma_{mag} = M_0$ at north.

$\sigma_{mag} \to$ monopole surface density!

Monopoles don't exist! $\to$ $\vec{\nabla}\bullet\vec{H} \neq \rho_{mag}$

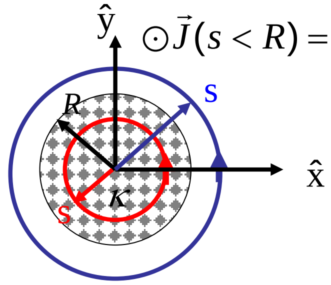
and $\vec{\nabla}\bullet\vec{H} \neq 0$ but $\vec{\nabla}\bullet\vec{B} = 0$ 7

A linear media is a material where B is proportional to H. The vacuum is a linear media where B is mu_0 times H. We can extend this to linear material by saying B = mu H: mu > mu_0 for paramagnetic material and mu < mu_0 for diamagnetic materials. The same thing happens for the electrical field. D is proportional to E with a proportionality constant of epsilon which usually exceeds epsilon_0 by a factor of Kappa. For historical reasons, the magnetic definitions are the reciprocal of the electric definitions. For the E-case the auxillary D field gets an additional factor epsilon from the physical E-field. For the B-case the physical field B appears to be scaled version of the auxillary H-field. As a result the magnetization is not proportional to kappa –1 times the B field but has a less transparent form.

Recalling the E-field case -- please do not think of the B-field as just a scaled version of H: such as thinking that B is just H in different units. Recall in the case of electrostatics the curl of E is zero but the curl of D might not be zero since the curl of the polarization might not vanish. In the case of magnetostatics, the divergence of B (ie del dot B) is zero, but the divergence of H may not be since there is no guarantee that the divergence of M is zero. To illustrate the perils of stretching the analogy, if we found a case where H had a non-zero divergence – we might conclude that there exists a free magnetic charge density in analogy with the divergence of E being proportional to the electric charge density rho. The uniformly magnetized sphere is an example where we have an infinite divergence on the surface of the sphere . We could find the sigma for these magnetic charges using the usual Gaussian pill box. We would mistakenly conclude that we found a surface *monopole* density consisting of isolated north or south poles or monopoles. There is no evidence for magnetic monopoles although they have been extensively and sensitively searched for. Hence the divergence of H can often be non-zero but the divergence of H has nothing to do with the monopole density.

The magnetic wire

$\odot \vec{J}(s<R) = J_0 \hat{z}$



Calculate bound currents from M

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} \rightarrow \vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H}$$

$$\vec{M} = \frac{\kappa_m J_0 s}{2}\hat{\phi} - \frac{J_0 s}{2}\hat{\phi} = \frac{\kappa - 1}{2} J_0 s\ \hat{\phi}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{\hat{z}}{s}\frac{\partial}{\partial s}\left(s M_\phi\right) = \frac{\kappa - 1}{2} J_0 \frac{\hat{z}}{s}\frac{\partial}{\partial s}\left(s^2\right) = (\kappa - 1) J_0 \hat{z}$$

$$\vec{K}_b(s=R) = \vec{M} \times \hat{s} = \frac{\kappa - 1}{2} J_0 R\ \hat{\phi} \times \hat{s} = -\frac{\kappa - 1}{2} J_0 R\ \hat{z}$$

Calculate H from free currents and then calculate B

$$\oint \vec{H} \bullet d\vec{\ell} = 2\pi s H_\phi = I_{enc} = \begin{matrix} J_0 \pi s^2 & s < R \\ J_0 \pi R^2 & s > R \end{matrix}$$

$$\vec{H}(s<R) = \frac{J_0 s}{2}\hat{\phi}\ ;\ \vec{H}(s>R) = \frac{J_0 R^2}{2s}\hat{\phi}$$

$$\vec{B}(s<R) = \frac{\kappa_m \mu_0 J_0 s}{2}\hat{\phi}\ ;\ \vec{B}(s>R) = \frac{\mu_0 J_0 R^2}{2s}\hat{\phi}$$

Calculate B from free and bound currents

$$J_{tot}(s<R) = J_0 \hat{z} + (\kappa - 1) J_0 \hat{z} = \kappa J_0 \hat{z}$$

$$2\pi s B_\phi = I_{enc} = \pi s^2 \kappa J_0 \rightarrow \vec{B}(s<R) = \frac{\kappa J_0 s}{2}\hat{\phi}$$

$$2\pi s B_\phi(s>R) = \mu_0 I_{enc} = \mu_0\left(\pi R^2 J_0 + I_b^{enc}\right)$$

$$I_b^{enc} = \int_S \vec{J}_b d\vec{a} + K_b \ell_\perp = \pi R^2 J_b + 2\pi R K_b =$$

$$I_b^{enc} = \pi R^2 (\kappa - 1) J_0 + 2\pi R\left(-\frac{\kappa - 1}{2} J_0 R\right) = 0$$

Thus $\vec{B}(s>R) = \dfrac{\mu_0 R^2 J_0}{2s}\hat{\phi}$

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Here we consider a thick wire which carries a uniform, free current density (J_0) in the z-direction. The wire is made of a magnetic material with a magnetic permeability of Kappa. Since the curl of H only depends on the free current, we can write a line integral expression for H_phi both inside and outside the wire. We can also find B from mu H . We can then compute the bound current density and surface current from the magnetization which can be found from B and H. The bound current density is given by the curl of M which is turns out to be (kappa -1) times the free current density. The surface current exists only at the radius R and points in the opposite direction as the free current. To complete the exercise we compute the B-field from the free and bound currents using the integral form of Ampere's law. For s < R , the bound surface current plays no role and the current densities just add to give Kappa times the free current density. Essentially Kappa acts as a current "multiplier" inside the material. This multiplier can be 10000 for ferromagnetic materials which **greatly** increases the magnetic field. For s > R, it is instructive to separate the enclosed current contributions into free and bound currents. Interestingly enough the total enclosed bound current vanishes with the surface current just canceling the integrated current density. This was inevitable since our expression for B outside the wire did not depend on mu but only on mu_o and would be the same expression we would get in the absence of bound currents. Recall the same thing happens in electrostatics in material. The total bound charge (in the form of rho_b and sigma_b) is zero which is easy to prove by the divergence theorem.

Boundary conditions

Electrostatic Boundary Conditions	Magnetostatic Boundary Conditions
$E_2^{\parallel} = E_1^{\parallel}$	$B_2^{\perp} = B_1^{\perp}$
$D_2^{\perp} - D_1^{\perp} = \sigma_{free}$	$H_2^{\parallel} - H_1^{\parallel} = \vec{K}_{free} \times \hat{n}$
$E_2^{\perp} - E_1^{\perp} = \frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$	$\vec{B}_2 - \vec{B}_1 = \mu_o\left(\vec{K}_{free} + \vec{K}_{Bound}\right) \times \hat{n}$
$D_2^{\parallel} - D_1^{\parallel} = P_2^{\parallel} - P_1^{\parallel}$	$H_2^{\perp} - H_1^{\perp} = -\left(M_2^{\perp} - M_1^{\perp}\right)$
$\varepsilon_2 \frac{\partial V_2}{\partial n}\Big\vert_{\text{interface}} - \varepsilon_1 \frac{\partial V_1}{\partial n}\Big\vert_{\text{interface}} = -\sigma_{free}$	$\frac{1}{\mu_2}\frac{\partial \vec{A}_2}{\partial n}\Big\vert_{\text{interface}} - \frac{1}{\mu_1}\frac{\partial \vec{A}_1}{\partial n}\Big\vert_{\text{interface}} = -\vec{K}_{free}$
$\varepsilon_0\left(\frac{\partial V_2}{\partial n}\Big\vert_{\text{interface}} - \frac{\partial V_1}{\partial n}\Big\vert_{\text{interface}}\right) = -\sigma_{TOT}$	$\frac{1}{\mu_o}\left(\frac{\partial \vec{A}_2}{\partial n}\Big\vert_{\text{interface}} - \frac{\partial \vec{A}_1}{\partial n}\Big\vert_{\text{interface}}\right) = -\vec{K}_{TOT}$

We note similar patterns in the electrostatic and magnetostatic boundary conditions.

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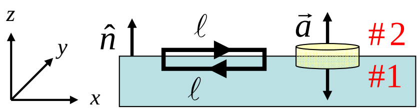
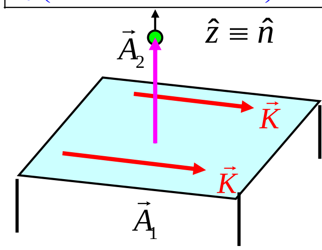
This slide summarizes some useful boundary conditions for solving magnetostatic and electrostatic problems involving materials. We have proved some of the electrostatic boundary conditions in the Polarize chp. We will prove some of the magnetostatic boundary conditions shortly and leave a few for homework. The 1,2 subscripts refer to being just inside or just outside the medium. The parallel and perpendicular are with respect to the medium surface. It is interesting to see how the magnetic boundary conditions complement the electric boundary conditions. For example, continuity of E-parallel across the boundary contrasts with continuity of B-perp. The BC follow from the vanishing of electric curl and the vanishing of the magnetic B divergence. We don't get E-perp continuity because of surface charge; we don't get B-parallel continuity because of surface currents. The next row makes this clear by showing the discontinuity conditions of the auxiliary D and H fields due to free (rather than bound) charges or currents. By way of contrast, the physical E and B fields have discontinuities related to both bound and free surface charge and currents. The next row highlights perils of thinking of the physical E and B fields as rescaled versions of the auxiliary D and H fields that we illustrated on the previous slide. Although the parallel component of E is continuous across a boundary, the parallel component of D is not if there is parallel polarization. Similarly the perpendicular components of B are continuous but the perpendicular components of H are not if there is perpendicular magnetization. We conclude by showing boundary conditions that apply to the electrical scalar potential or the magnetic vector potential. Here n is the coordinate transverse to the surface. In several places you can see how a 1/epsilon factor in an electric BC converts to a mu factor in a magnetic BC.

Two boundary condition "proofs"

$$\boxed{\vec{B}_2 - \vec{B}_1 = \mu_o\left(\vec{K}_{free} + \vec{K}_{Bound}\right)\times\hat{n}}$$

$$\boxed{\frac{1}{\mu_o}\left(\frac{\partial \vec{A}_2}{\partial n}\Big|_{\text{interface}} - \frac{\partial \vec{A}_1}{\partial n}\Big|_{\text{interface}}\right) = -\vec{K}_{TOT}}$$





Define $\hat{y} \parallel \left(\vec{K}_b + \vec{K}_f\right)$, $\hat{z} \parallel \hat{\eta}$, $\hat{x} \parallel \vec{K}\times\hat{\eta}$

$$\vec{B}_2\cdot\vec{a} - \vec{B}_1\cdot\vec{a} = \int_V \nabla\cdot \mathrm{B}\, d\tau = 0 \to \boxed{\vec{B}_2^z = \vec{B}_1^z}$$

$$\oint \mathrm{B}\cdot d\ell = \left(\vec{B}_2^x - \vec{B}_1^x\right)\ell = \int_S \nabla\times\mathrm{B}\cdot d\vec{a} = \mu_0 I_{\text{inc}}$$

$$= \mu_0\left(K_b + K_f\right)\ell \to \vec{B}_2^x - \vec{B}_1^x = \mu_0\left|\vec{K}_b + \vec{K}_f\right|$$

$\vec{B}_2^y - \vec{B}_1^y = 0$ since no K_x.

$$\boxed{\vec{B}_2 - \vec{B}_1 = \mu_0\left|\vec{K}_b + \vec{K}_f\right|\hat{x} = \mu_0\left(\vec{K}_b + \vec{K}_f\right)\times\hat{\eta}}$$

$$\vec{A}(z) = \left[-\frac{\mu_0\vec{K}\left(|z| - \infty\right)}{2}\right]_{\text{current sheet}} + \vec{A}_{\text{continuous}}$$

$$\frac{\partial \vec{A}_2}{\partial z} = -\frac{\partial}{\partial z}\frac{\mu_0\vec{K}\left[(+z) - \infty\right]}{2} = -\frac{\mu_0\vec{K}}{2}$$

$$\frac{\partial \vec{A}_1}{\partial z} = -\frac{\partial}{\partial z}\frac{\mu_0\vec{K}\left[(-z) - \infty\right]}{2} = +\frac{\mu_0\vec{K}}{2}$$

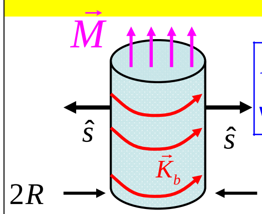
$$\frac{\partial \vec{A}_2}{\partial z} - \frac{\partial \vec{A}_1}{\partial z} = -\mu_0\vec{K}\ ;\ \vec{K} = \vec{K}_f + \vec{K}_b$$

10

b

On the left we give the two BC for the magnetic field on either side of a material. The first of these says that B-perp is continuous across the surface. This follows from the fact that there are no magnetic monopoles. This means that the net magnetic charge enclosed in the Gaussian pill box on the surface is zero which means the flux through the upper face always cancels the flux on the lower face. The second BC implied by the B expression is a discontinuity in the B-parallel components due to surface currents – in this case the bound plus the free current since B is sensitive to both. The normal out of the surface points along z. The y-axis is defined as the direction of the total surface current. The x-components of B are continuous since (by definition) the K_tot is in the y direction and hence K_tot does not pierce Ampere's loop in the y-z plane. The y-component of B does pierce Ampere's loop in the z-x plane creating a discontinuity in Bx. We can write x-hat in terms of the cross product of Ktot and the normal n. The right side BC is for the magnetic vector potential. We assume that the boundary at z=0 carries a surface curret vec-K which might consist of free as well as bound contributions. The surface current will be the source of discontinuity in B or in A. If we are very close to the boundary carrying the surface current, our boundary will look like an infinite current sheet. We can thus simply recycle the result we obtained in the magnetostatics chapter for the vector potential on either side of a current sheet. Of course there may be a slowly varying A contribution representing a continuous B-field which implies continuous A derivatives. Here our "normal" coordinate is just z which we add in. Since our current sheet expression depends on |z| the z dependence or partial A/partial z will change sign going from +z to –z. Here vec-A2 is for positive z and vec-A1 is for –z. We construct these derivatives for the current sheet contribution and subtract them to obtain our result. Presumably the partial A_continuous / partial z will be continuous across the boundary and thus cancel in the difference.

A' boundary condition for magnetized cylinder



$\vec{B}(s<R)=\mu_0 M_0 \hat{z}$
with $\vec{K}_b$=$M_0\hat{\phi}$

$\vec{A}(s<R)=\frac{\mu_0 M_0 s}{2}\hat{\phi}$; $\vec{A}(s>R)=\frac{\mu_0 M_0 R^2}{2s}\hat{\phi}$

In both cases $\vec{\nabla}\bullet\vec{A}=0$ so BC can be applied

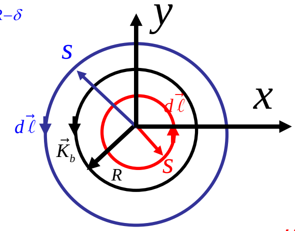
$$\left[\frac{\partial \vec{A}}{\partial s}\right]_{R+\delta}=\frac{\mu_0 M_0 R^2\hat{\phi}}{2}\left[\frac{\partial}{\partial s}\frac{1}{s}\right]_R=-\frac{\mu_0 M_0\hat{\phi}}{2}$$

$$\left[\frac{\partial \vec{A}}{\partial s}\right]_{R-\delta}=\frac{\mu_0 M_0\hat{\phi}}{2}\left[\frac{\partial s}{\partial s}\right]_R=\frac{\mu_0 M_0\hat{\phi}}{2}$$

$$\left[\frac{\partial \vec{A}}{\partial s}\right]_{R+\delta}-\left[\frac{\partial \vec{A}}{\partial s}\right]_{R-\delta}=-\mu_0 M_0\hat{\phi}=-\mu_0\vec{K}_b$$

Everything checks!

$$\left[\frac{\partial \vec{A}}{\partial s}\right]_{R+\delta}-\left[\frac{\partial \vec{A}}{\partial s}\right]_{R-\delta}=-\mu_0\vec{K}=-\mu_0 M_0\hat{\phi}$$



For s < R

$$\Phi_m=\mu_0 M_0\pi s^2 \rightarrow A_\phi 2\pi s=\mu_0 M_0\pi s^2 \rightarrow \frac{\mu_0 M_0\pi s^2}{2\pi s}$$

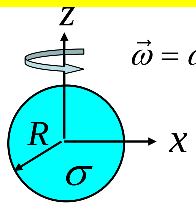
For s > R

$$\Phi_m=M_0\pi R^2 \rightarrow A_\phi 2\pi s=\mu_0 M_0\pi s^2 \rightarrow \frac{\mu_0 M_0\pi R^2}{2\pi s}$$

11

As an illustration of the vector potential derivative discontinuity boundary condition, we consider the infinite magnetized cylinder aligned on the z-axis. Here the bound surface current is in the M_0 phi-hat direction. We work in cylindrical coordinates. We compute the vector potential using the flux method both inside and outside the cylinder radius R and get expressions for A(s>R) and A(s<R). We can verify from these forms that A is continuous at the boundary s = R. Our A derivative discontinuity condition is only valid if we are in the Coulomb gauge (or the gauge where the divergence of A vanishes), but this is the case for our A expressions since there is no derivative in the direction of the field (eg no phi dependence for A_phi). The normal direction for the cylinder is the s direction. So we take derivatives of our A expressions with respect to s and evaluate these derivatives at s=R. The difference of the derivatives is the negative of mu_0 M_0 phi-hat which is the negative of –mu_0 times the surface current which confirms our A – slope discontinuity boundary condition.

A Boundary Conditions for Spinning Sphere



$\vec{\omega} = \omega\hat{z}$

$$\vec{A}_1\left(|\vec{r}| < R\right) = \frac{\mu_0 R\sigma}{3}\vec{\omega}\times\vec{r}$$

$$\vec{A}_2\left(|\vec{r}| > R\right) = \frac{\mu_0 R^4\sigma}{3r^3}\vec{\omega}\times\vec{r}$$

Note $\vec{A}$ is continuous at $r = R$

$$\boxed{\vec{A}_1(R) = \frac{\mu_0 R\sigma}{3}\vec{\omega}\times\hat{r} = \vec{A}_2(R)}$$

Check discontinuity BC

$$\frac{1}{\mu_o}\left(\frac{\partial\vec{A}_2}{\partial n}\Big|_{\text{interface}} - \frac{\partial\vec{A}_1}{\partial n}\Big|_{\text{interface}}\right) = -\vec{K}_{TOT}$$

Here $n \to r$ and

$$\vec{K}_{TOT} = \vec{K}_{\text{free}} = \sigma\vec{\omega}\times R\hat{r} = \sigma\omega R\sin\theta\,\hat{\phi}$$

$$\frac{1}{\mu_o}\left(\frac{\partial\vec{A}_2}{\partial r}\Big|_R - \frac{\partial\vec{A}_1}{\partial r}\Big|_R\right) = -\sigma\omega R\sin\theta\,\hat{\phi}$$

$$\vec{\omega}\times\vec{r} = r\omega\sin\theta\,\hat{\phi} \to$$

$$\vec{A}_1 = \frac{\mu_0 R\sigma\omega}{3} r\sin\theta\,\hat{\phi} \;\&\; \vec{A}_2 = \frac{\mu_0 R^4\sigma\omega}{3r^2}\sin\theta\,\hat{\phi}$$

$$\frac{1}{\mu_o}\left(\frac{\partial\vec{A}_2}{\partial r}\Big|_R - \frac{\partial\vec{A}_1}{\partial r}\Big|_R\right)$$

$$= \frac{1}{\mu_o}\left(\frac{\partial}{\partial r}\left\{\frac{\mu_0 R^4\sigma\omega}{3r^2}\sin\theta\,\hat{\phi}\right\} - \frac{\partial}{\partial r}\left\{\frac{\mu_0 R\sigma\omega}{3} r\sin\theta\,\hat{\phi}\right\}\right)_R$$

$$\frac{1}{\mu_o}\left(\frac{-2\mu_0 R^4\sigma\omega}{3r^3}\sin\theta\,\hat{\phi} - \frac{\mu_0 R\sigma\omega}{3}\sin\theta\,\hat{\phi}\right)_R$$

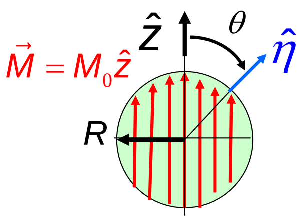
$$\boxed{\therefore -\frac{1}{\mu_o}\mu_0 R\sigma\omega\sin\theta\,\hat{\phi} = -\sigma\omega R\sin\theta\,\hat{\phi} \text{ (checks!)}}$$

12

To flesh out our discussion of the BC, we illustrate the vector potential BC for the case of the spinning ball of charge. We being by reviewing the A-solutions for inside and outside the sphere. The first thing we should check is continuity of the vector potential itself. Although surface currents create a discontinuity in the derivative of A – A itself is continuous. The same comment applies to the electric scalar potential V. Here we just put r=R and show A1(r=R) = A2(r=R).

We next check the derivative discontinuity BC for A. Here the normal vector is just r-hat which points radially outwards. The derivative discontinuity is proportional to the total surface current – in this case K is free current, while in the magnetized sphere it is a bound current. Both the vector potential and the surface current is in the phi direction. We begin by casting omega cross r in terms of r and theta. Here r is radial distance to the point at which we are evaluating A. We next take the derivatives of A1 and A2 with respect to r and take the derivative difference evaluated on the surface (r = R). Indeed we find the derivative discontinuity is given by the BC and is essentially the negative of the surface current.

Lets also check the magnetized sphere



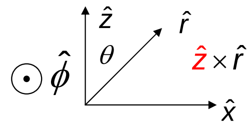
$$\vec{A}_1 = \frac{\mu_0 R\sigma\omega}{3} r\sin\theta\,\hat{\phi}$$

$$\vec{A}_2 = \frac{\mu_0 R^4\sigma\omega}{3r^2}\sin\theta\,\hat{\phi}$$

$\sigma\,\omega\,R \to M_0$ then:

$$\vec{A}(|\vec{r}| < R) = \frac{\mu_0 R}{3} M_0 r\sin\theta\,\hat{\phi}$$

$$\vec{A}(|\vec{r}| > R) = \frac{\mu_0 R^4\sigma}{3r^3} M_0\sin\theta\,\hat{\phi}$$



$\hat{z}\times\hat{r} = \sin\theta\,\hat{\phi}$

$$\frac{1}{\mu_o}\left(\frac{\partial\vec{A}_2}{\partial r}\Big|_R - \frac{\partial\vec{A}_1}{\partial r}\Big|_R\right) = -\vec{K}$$

For the spinning sphere

$$\frac{1}{\mu_o}\left(\frac{\partial\vec{A}_2}{\partial r}\Big|_R - \frac{\partial\vec{A}_1}{\partial r}\Big|_R\right) = -\sigma\omega R\sin\theta\,\hat{\phi}$$

Substituting $\sigma\omega R \to M_0$ gives us

$$\frac{1}{\mu_o}\left(\frac{\partial\vec{A}_2}{\partial r}\Big|_R - \frac{\partial\vec{A}_1}{\partial r}\Big|_R\right) = -M_0\sin\theta\,\hat{\phi}$$

Hence we expect $M_0\sin\theta\,\hat{\phi} = \mu_0\vec{K}$

$\vec{K}_b = \vec{M}\times\hat{n} = M_0\,\hat{z}\times\hat{r}$

$\to M_0\sin\theta\,\hat{\phi} = \vec{K}_b = \vec{K}$

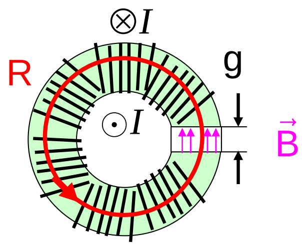
13

Not surprisingly the vector potential derivative discontinuity also works for the uniformly magnetized sphere. We borrow the expressions for A1 and A2 and replace sigma omega R with M0. We can also apply the same substitution to the derivative difference for the spinning ball of charge. If the BC holds we have an expected form for the surface current K.

Indeed this is the same surface current obtained from M cross r-hat.

A gapped magnet

Assume N turns
with current I and $g \ll R$



$$\oint \vec{H} \bullet d\vec{\ell} = NI$$

$$B_2^{\perp} = B_1^{\perp} \rightarrow B_{gap} \approx B$$

$$\frac{B}{\mu}(2\pi R) + \frac{B}{\mu_0} g \approx NI$$

$$B \approx \frac{NI\mu}{2\pi R + \frac{\mu}{\mu_0} g} \approx \frac{NI\mu}{2\pi R}\left(\frac{1}{1 + \frac{\mu}{\mu_0}\frac{g}{2\pi R}}\right)$$

Hence $\dfrac{B_{\text{w/ gap}}}{B_{\text{no gap}}} \approx \dfrac{1}{1 + \frac{\mu}{\mu_0}\frac{g}{2\pi R}}$

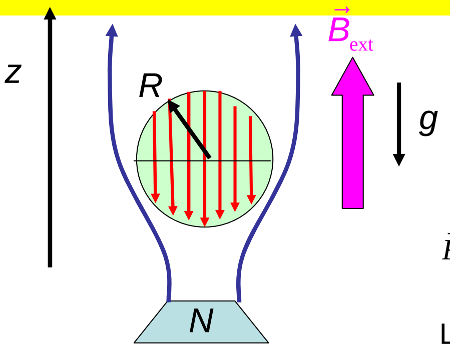
Ferromags can have $\kappa_m = \mu/\mu_0 > 10000$

Hence even $\dfrac{g}{2\pi R} = 1\%$ can
reduce B by a factor of over 100
and require $100\times$ current for same
B-field as required for an ungapped magnet.

14

Here is a practical application of magnetostatics. It is an idealized version of a gapped, conventional analysis magnet. In my business – experimental high energy physics – these magnets are used to reconstruct the momentum of subatomic particles created at accelerators by measuring their bends through the magnetic field (typically about ½ Tesla). To get to such large B-fields we use ferromagnetic materials such as steel which we model as a material with a large kappa_m = mu/mu_0 ratio. It is crucial that we include an air gap to avoid having the subatomic particles scattering in the magnet steel. We calculate the magnetic field using Ampere's law for H around a loop passing through the center of the toroidal magnet. The current enclosed by the loop is given by the amp-turns or the current carried by the wire times the number of turns wound around the magnet. We next write the H-line integral in terms of the the B-field we wish to calculate. If the gap is small compared to the radius we get a line integral of B/mu while in the steel with a path of essentially 2 pi R and a gap (g) contribution of B/mu_o . For a small gap, the B-field in the gap should very close to the B-field in the magnet steel because of B-perp continuity. We can solve for the B-field and write the B field for the gapped magnet in terms of the B-field we would get if the magnet steel was continuous with no gap. The ratio depends on kappa_m and the gap fraction of the magnet. Good ferromagnets can have kappa_m in excess of 10,000. This means if the gap length is only 1% of the full Amperian loop, the B-field of the magnet will be 100 times smaller than that of an ungapped magnet running the same amp-turns. Since the subatomic particles in high energy physics have considerable momenta, a large B-field is required to get a well measured deflection hence a very large number of amp-turns is required.

Superconductor levitation



Meissner effect: A superconductor expels $\vec{B}$ completely

$$\vec{B}_{in} = \frac{2\mu_0 \vec{M}}{3} + \vec{B}_{ext} = \vec{0}$$

$$\vec{F} = \nabla \vec{m} \bullet \vec{B} = \frac{\partial mB}{\partial z}\hat{z} = -\frac{2\pi R^3}{\mu_0}\frac{\partial B^2}{\partial z}\hat{z}$$

Levitate if $-\frac{2\pi R^3}{\mu_0}\frac{\partial B^2}{\partial z}\hat{z} - \frac{4\pi R^3 \rho_{mass} g\, \hat{z}}{3} = 0$

$$\frac{\partial B^2}{\partial z} = -\left[\frac{2\mu_0 \rho_{mass} g}{3}\right]$$

System "acts" like a perfect diamagnet with

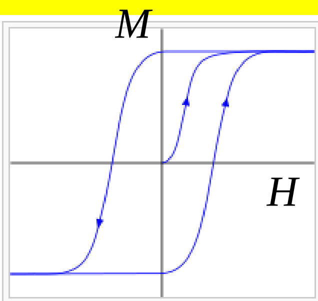
$$\vec{M} = -\frac{3\vec{B}_{ext}}{2\mu_0} \; ; \; \vec{m} = \frac{-2\pi R^3 \vec{B}_{ext}}{\mu_0}$$

In reality free currents --not bound currents--are responsible for Meissner effect.

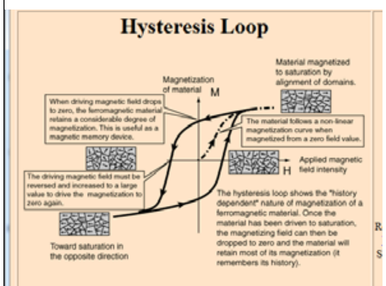
This is a <u>rough</u> estimate which requires a reasonably constant B in the sphere volume. 15

With the advent of high Tc superconductors, one can get superconduction with liquid nitrogen and can demonstrate superconductor properties in a science kit made for children. One demonstration which you may have seen is superconductor magnetic levitation. The same principle on a larger scale can be used to levitate trains! Levitation is based on the Meissner effect which says that a superconductor creates surface currents to expel all of the B-field from the superconductor volume. These are actual free currents but I think I can model them by the same surface currents present in the spinning charge ball which are the same as those created by a uniform magnetization in a superconducting sphere. In the magnetized sphere, the M-vector creates a B-field given by 2/3 mu_0 times the magnetization M. To get the Meissner effect superconductor creates a magnetization that exactly cancels the applied B-field and hence no magnetic field exists in the sphere. The total magnetic moment of the sphere will be this magnetization times the volume. The total dipole moment can then interact with the external magnetic field which induced the dipole moment. The force will be proportional to the gradient of the dipole potential energy. Let us assume that the external field is provided by a conventional magnet placed below the sphere. Since the dipole moment opposes this external field we get a minus sign for the force. Since the external field will die off as a function of the height z, the derivative of the external field is negative which means the force points up and can balance the weight of the sphere. Both the weight and total magnetic dipole moment are proportional to the sphere volume so the levitation condition should be independent of the sphere size. We can use either a north or south pole for the external field since the levitation condition is proportional to B times dB/dz so it is independent of the sign of B. The sphere will raise to a height related to the gradient.

Hysteresis



Hysteresis loop: magnetization (M) as function of magnetic field strength (H)



Ferromagnets have a very non-linear M versus H curve indicating a complicated magnetization mechanism.
The curve is not even reproducable but depends on the history of how H was applied $\rightarrow$ hysteresis
In this example, as we raise H from 0, M reaches saturation.
The saturated M remains as H is lowered to 0.
If one makes a new cycle the M versus H curve changes.

The very high μ and hysteresis is due to domain formation. Domains are pieces of the material with locally parallel magnetizations.
The "best" ferromagnets are often not iron.

16

Ferromagnetic materials are highly non-linear and exhibit an important phenomena called hysteresis. The main idea is that there is no universal M versus B curve for ferromagnetic materials but M versus B depends on exactly how the material is energized and its past history. The illustrated curve shows hysteresis as well as magnetic saturation. At some point all the relevant dipoles in a ferromagnet are aligned and one sees no increase in M with increasing H (eg more current). The physics behind ferromagnetism is complicated. The basic idea is the formation of magnetic domains when the magnetic moments within a domain are highly correlated. Hysteresis is technologically important since it leads to permanent magnets and was used in early magnetic memory devices for computers. High field permanent magnets are often not made of iron. One example is alnico magnets made of aluminum, nickel and cobalt.