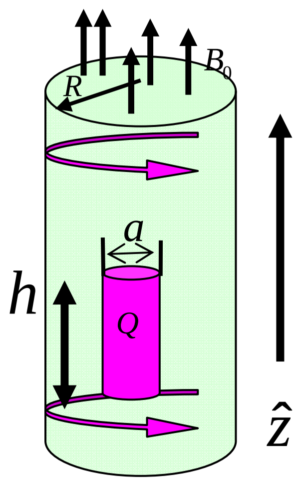
Jim Wiss
Jan 6, 2011

A "Maxwell Cylinder" Paradox

The Maxwell cylinder system discussed in the Conservation lecture notes consists of two oppositely charged cylindrical shells immersed in a long, solenoidal magnetic field which rotates as the $\vec{B}$ field is turned off and the shells absorb the angular momentum stored in the fields. Our treatment was similar to Griffiths ex 8.4. Here I consider angular momentum balance by a single cylinder of charge Q, radius a, inside of a solenoid of radius R with a $\vec{B} = B_0\hat{z}$.



This analysis will explain why the angular momentum balance exercises confine $\vec{E}$ to a small region to avoid fringe field complications.

Computing $\vec{L}$ from Faraday's Law

This is essentially the same argument as in the lecture notes

We have long cylinder w/ +Q within solenoid

Calculate the torque on +Q as $\vec{B}$ changes.

$$\varepsilon = -\frac{\partial \Phi}{\partial t} = -\frac{\partial\left(\pi a^2 B_z\right)}{\partial t} = -\left(\pi a^2\right)\frac{\partial B_z}{\partial t} = \oint \vec{E} \bullet d\vec{\ell} = 2\pi a E_\phi$$

$$E_\phi = -\frac{a^2}{2a}\frac{\partial B_z}{\partial t} \;;\; F_\phi = QE_\phi = -\frac{a^2 Q}{2a}\frac{\partial B_z}{\partial t}$$

$$\vec{N}_a = \vec{r} \times \vec{F} = -a\hat{s} \times \frac{Qa^2}{2a}\frac{\partial B_z}{\partial t}\hat{\phi} = -\frac{Qa^2}{2}\frac{\partial B_z}{\partial t}\hat{z}$$

Start with $\vec{B}(0) = B_0\hat{z}$ and slowly turn off B: $\vec{B}(\infty) = 0$

$$\vec{L}_{FL} = \int_0^\infty \vec{N} dt = -\frac{Qa^2}{2}\hat{z}\int_{B_0}^{0} dB' = +\frac{Qa^2 B_0}{2}\hat{z}$$

A field angular momentum paradox

We now compute $\vec{L}$ stored in the magnetic fields for the one cylinder system.

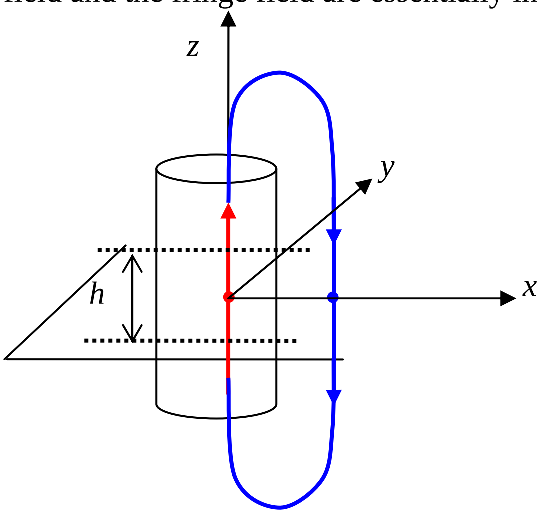
$$\vec{\ell}\left(a < s < R\right) = \vec{r} \times \vec{\wp} = s\hat{s} \times \left[\varepsilon_0\left(\frac{Q\hat{s}}{2\pi\varepsilon_0 h s}\right) \times \left(B_0\hat{z}\right)\right]_{\wp} = \frac{-QB_0\hat{z}}{2\pi h}$$

Our $\vec{\ell}$ is non-zero only in regions where there are both electric and magnetic fields which for the infinite solenoid and cylinder would be $a < s < R$.

Hence $\vec{L}_{\text{field}} = \int \vec{\ell} d\tau = \frac{-QB_0\hat{z}}{2\pi h}\left[\pi\left(R^2 - a^2\right)h\right]_\tau = \frac{-QB_0\left(R^2-a^2\right)\hat{z}}{2} \neq \left[\frac{QB_0 a^2\hat{z}}{2}\right]_{\text{LF}}$

<u>Resolving the paradox with fringe fields</u>

The problem with our field calculation is that any finite length solenoid will have a fringe field and we need to include the fringe field contribution. We use the below simplified geometry for the fringe fields. We are assuming that the charged cylinder is centered on the solenoid and has a height h which sufficiently short so that both the inside solenoidal field and the fringe field are essentially in the $\pm\hat{z}$ direction.



Here is how we can include the fringe field in our $\vec{L}_{\text{field}}$ calculation.

$$\vec{\ell}_{\text{fringe}} = s\hat{s}\times\left[\varepsilon_0\left(\frac{Q\hat{s}}{2\pi\varepsilon_0 hs}\right)\times\left(B_z(s)\hat{z}\right)\right]_{\not p} = \frac{-Q\hat{z}}{2\pi h}B_z(s) \to \vec{L}_{\text{fringe}} = \frac{-Q\hat{z}}{2\pi}\int_R^\infty 2\pi s ds B_z(s)$$

We note that $B_z(s>R) < 0$ in our model and that we don't know at this point what the actual $B_z(s)$ function is. However we do know that

$$\int_{\text{z=0 plane}} B\bullet d\vec{a} = 0 = \int_0^R 2\pi s ds B_0 + \int_R^\infty 2\pi s ds B_z(s) \to \int_R^\infty 2\pi s ds B_z(s) = -\pi R^2 B_0$$

This is because $\vec{\nabla}\bullet\vec{B} = 0 \to \int_{z>0} \vec{B}\bullet d\vec{a} = 0$ where we can think of the z=0 plane as a bounding surface. We would thus have

$$\vec{L}_{\text{fringe}} = \frac{-Q\hat{z}}{2\pi}\int_R^\infty 2\pi s ds B_z(s) = \frac{-Q\hat{z}}{2\pi}\left(-\pi R^2\right)$$

$$\to \vec{L}_{\text{fringe}} + \vec{L}_{s<R} = \frac{-Q\hat{z}}{2\pi}\left(-\pi R^2\right) + \frac{-QB_0\left(R^2-a^2\right)\hat{z}}{2} = \frac{QB_0 a^2\hat{z}}{2} = \vec{L}_{\text{FL}}$$

and the paradox is resolved!

Hence with this simple-to-calculate geometry we get a field angular momentum consistent w/ Faraday's law even with one cylinder once fringe fields are considered.