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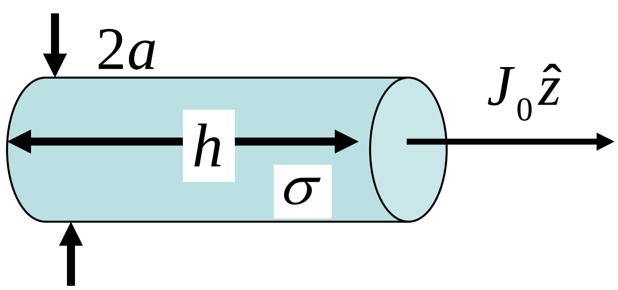
Resistor Poynting Power Paradox

Abstract

We discuss the use of Poynting's theorem to obtain the power converted into heat for a cylindrical resistor with a static, uniform current within the resistor and zero current and E-field outside of the resistor. We show that Poynting's theorem works for cylindrical bounding surfaces contained *within* the resistor but fails for surfaces which *enclose* the resistor. The problem is that our model E-field has a non-zero curl in the $\hat{\phi}$ direction at the resistor surface. We show how Poynting's theorem can be resurrected by adding a singular, artificial time variation in the magnetic field at the resistor surface. We conclude with a discussion of a similar paradox when one uses the Maxwell stress tensor to calculate the pressure on the resistor surface.

Creating the Paradox

Consider applying Poynting's theorem to a simple cylindrical resistor of radius a, length $h \gg a$ with uniform current density $\vec{J} = J_0 \hat{z}$ and conductivity. σ,



The differential form Poynting's theorem is $\vec{\nabla} \bullet \vec{S} + \frac{\partial u_{EM}}{\partial t} + \vec{E} \bullet \vec{J} = 0$. Integrating over volume and using the divergence theorem gives us the integral form

$\int_{\mathcal{S} \subseteq \mathcal{V}} \vec{S} \bullet d\vec{a} + \cancel{\frac{\partial U_{EM}}{\partial t}} + \int_{\mathcal{V}} \vec{E} \bullet \vec{J} \; d\tau = 0$ where $\mathcal{S}$ is any surface which "encloses" $\mathcal{V}$ and we dropped $\partial U_{EM} / \partial t$ contribution since the fields are static in our model. The Poynting vector is given by

$$\vec{S} = \frac{\vec{E} \times B}{\mu_0} = \frac{1}{\mu_0} \left[\frac{J_0}{\sigma} \hat{z} \right]_{\vec{E}} \times \left[\frac{\mu_0 s J_0 \hat{\phi}}{2} \right]_{\vec{B}} = \begin{cases} -\dfrac{s J_0^2 \hat{s}}{2\sigma} & \text{for } s < a \\ 0 & \text{for } s > a \end{cases}$$

which disappears outside of the resistor since $\vec{E}$ disappears. If we apply the integral form to a bounding surface of length h and a radius R within the resistor ($R < a$) we get:

$$\int_{S\subseteq \mathcal{V}} \vec{S}\bullet d\vec{a} + \frac{\partial U_{EM}}{\partial t} + \int_{\mathcal{V}} \vec{E}\bullet\vec{J}\; d\tau = \left[-\frac{RJ_0^2\hat{s}}{2\sigma}\bullet\left(2\pi Rh\,\hat{s}\right)\right]_{\int\limits_{S\subseteq\mathcal{V}} \vec{S}\bullet d\vec{a}} + \left[0\right]_{\frac{\partial U_{EM}}{\partial t}} + \left[\left(\frac{J_0\hat{z}}{\sigma}\bullet J_0\hat{z}\right)_{\vec{E}\bullet\vec{J}} \left(\pi R^2 h\right)_\tau\right]_{\int\limits_{\mathcal{V}}\vec{E}\bullet\vec{J}\;d\tau}$$

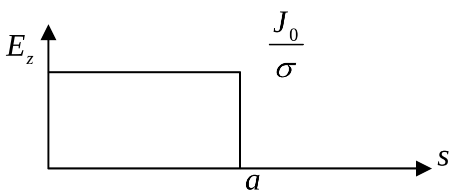
$$\frac{-\pi R^2 h J_0^2}{\sigma} + \frac{\pi R^2 h J_0^2}{\sigma} = 0 \text{ for } R < a$$

The paradox comes about when we use an enclosing bounding surface with $R > a$. $\int_{\mathcal{V}} \vec{E}\bullet\vec{J}\; d\tau$ is $\pi a^2 h J_0^2/\sigma$ but now $\int_{S\subseteq\mathcal{V}} \vec{S}\bullet d\vec{a}$ =0 since $\vec{S}(s>a)=0$ which violates our integral version of Poynting's theorem. What went wrong?

Removing the Paradox

The problem is that our static resistor model violates Faraday's law which is necessary to prove Poynting's theorem.

Here is a plot of the electrical field as a function of s in our model.



$$\vec{E} = \hat{z}\frac{J_0}{\sigma}\left[1-\Theta(s-a)\right]$$

where we introduce the Heaviside function $\Theta(s-a) = \begin{matrix} 1 \text{ if } s > a \\ 0 \text{ if } s < a \end{matrix}$. The derivative of the Heaviside function is a delta function implying a non-zero curl for the electric field.

(1.1) $$\vec{\nabla}\times\vec{E} = -\hat{\phi}\frac{\partial E_z}{\partial s} = -\hat{\phi}\frac{\partial}{\partial s}\left\{\frac{J_0}{\sigma}\left[1-\Theta(s-a)\right]\right\} = \hat{\phi}\frac{J_0}{\sigma}\delta(s-a) = -\frac{\partial\vec{B}}{\partial t}$$

This means our model is not correct for a static current resistor. We can still check Poynting's theorem, however, by including the singular $\partial U_{EM}/\partial t$ term due to $\partial\vec{B}/\partial t$ once the volume includes $s=a$ which would be true when $R>a$.

$$\frac{\partial U_{EM}}{\partial t} = \int\frac{\partial u}{\partial t}\left[2\pi hsds\right]_{d\tau}\;;\; \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left\{\frac{B_\phi^2}{2\mu_0}\right\} = \frac{B_\phi(s)}{\mu_0}\frac{\partial B_\phi}{\partial t} = \left[\frac{J_0 s}{2}\right]_{\frac{B}{\mu_0}}\left[-\frac{J_0}{\sigma}\delta(s-a)\right]_{\frac{\partial B_\phi}{\partial t}}$$

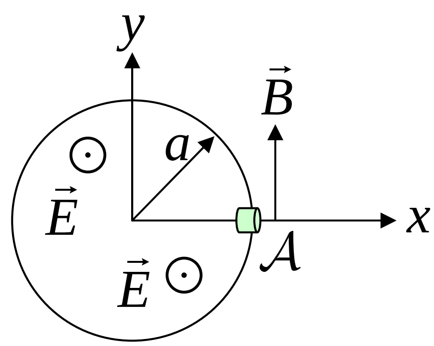
$$\frac{\partial u}{\partial t} = -\frac{s\,J_0^2}{2\sigma}\delta(s-a) \rightarrow \frac{\partial U_{EM}}{\partial t} = \int\frac{\partial u}{\partial t}2\pi hsds = 2\pi h\int s^2 ds\left[-\frac{J_0^2}{2\sigma}\delta(s-a)\right]_{\frac{\partial u}{\partial t}} = -\frac{J_0^2}{\sigma}\pi a^2 h$$

Let us write the integral form of Poynting's theorem for the case where $R>a$:

$$\int\vec{S}\bullet d\vec{a} + \frac{\partial U_{EM}}{\partial t} + \int_{\mathcal{V}}\vec{E}\bullet\vec{J}\; d\tau = 0 \rightarrow \left[0\right]_{\int\vec{S}\bullet d\vec{a}} + \left[-\frac{J_0^2\pi a^2 h}{\sigma}\right]_{\frac{\partial U_{EM}}{\partial t}} + \left[\frac{\pi a^2 h J_0^2}{\sigma}\right]_{\int\limits_{\mathcal{V}}\vec{E}\bullet\vec{J}\;d\tau} = 0$$

The Poynting vector surface integral disappears for an enclosing bounding cylinder ($R > a$) , and the rate of change of the E&M model cancels the $\vec{E} \bullet \vec{J}$ integral to restore Poynting's theorem .Thus Poynting's Theorem is still valid once we compensate for the Faraday's law violation by adding an artificial, singular $\partial \vec{B}_\phi / \partial t$ at the resistor surface.

A Related Stress Tensor Paradox



We start with the wrong approach that leads to a paradox. We use the stress tensor to calculate the pressure on the resistor surface assuming static $\vec{E}$ and $\vec{B}$ fields. We will calculate the pressure using a infinitesimal cylinder with end plane area $\mathcal{A}$ which extends just within and just outside of the resistor centered at $\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} a & 0 \end{pmatrix}$. The force due to the currents within the cylinder will be

$$\vec{F}_x = T_{xx}^{>}\mathcal{A} - T_{xx}^{<}\mathcal{A} = -\frac{\varepsilon_0 E_z^2}{2}\left(-\mathcal{A}\right) = \frac{\varepsilon_0 J_0^2}{2\sigma^2}\mathcal{A} \rightarrow \mathcal{P}_{\text{outward}} = \frac{\varepsilon_0 J_0^2}{2\sigma^2} \quad (\text{wrong!})$$

We only considered the electrical stress tensor since it is the only tensor contribution which is discontinuous across the resistor boundary. I believe this answer is wrong for the following reasons.

1. The pressure in this wrong treatment is evidently due to the electrical field because it is obtained from the electrical stress tensor and involves σ which only affects the electric field. The conductivity σ is irrelevant to the magnetic field.
2. The electric field is parallel to the resistor surface and would not contribute to a force normal to the cylindrical surface.
3. There are no free charges anywhere for the electrical fields to act on.
4. One could expect a magnetic pressure contribution of the form: $\mathcal{P} = \vec{K} \times \left(\frac{\vec{B}_{>} + \vec{B}_{<}}{2} \right)$ where we use the average of the field on either side of the boundary to get the magnetic field not due to the surface current $\vec{K}$. But there is only a finite $\vec{J}$ and no surface current $\vec{K}$. This is reinforced by noting that $\vec{B}$ .is continuous across the resistor surface.
5. Two parallel magnetic currents would attract implying a negative outward pressure, but our answer suggests a current repulsion.

The foregoing suggests that the correct answer must be $\mathcal{P} = 0$ on the surface.

Removing the Stress Tensor Paradox

We will calculate the pressure on the surface using the differential form of the stress tensor equation:

$$\vec{f} = \vec{\nabla}\bullet\vec{T} - \varepsilon_0\mu_0\frac{\partial\vec{S}}{\partial t} \to \vec{f}_x^{\text{singular}} = \frac{\partial T_{xx}^{\text{elect}}}{\partial x} - \varepsilon_0\mu_0\frac{\partial\vec{S}_x}{\partial t}\ ;\ \mathcal{P}_{\text{outward}} = \lim_{\delta\to 0}\int_{a-\delta}^{a+\delta}\vec{f}_x^{\text{singular}}dx$$

In order to affect the pressure on the resistor surface we are looking for singular parts of the force density since the surface force is the integral of force density over the infinitesimal cylinder. We use T_{xx}^{elect} since this the only singular part of $\vec{\nabla}\bullet\vec{T}$. The magnetic stress tensor is continuous. We next calculate the singular parts of the electric stress tensor and the rate of change of the field momentum. The stress tensor discontinuity is due to the discontinuity in E_z which is discussed in Eq. (1.1)

$$\frac{\partial T_{xx}^{\text{elect}}}{\partial x} = -\frac{\partial}{\partial x}\left\{\frac{\varepsilon_0 E_z^2}{2}\right\} = -\varepsilon_0 E_z\frac{\partial E_z}{\partial x}\xrightarrow{x=s} -\varepsilon_0\frac{J_0}{\sigma}\frac{\partial}{\partial x}\left\{\frac{J_0}{\sigma}\left[1-\Theta\left(x-a\right)\right]\right\} = \frac{\varepsilon_0 J_0^2}{\sigma^2}\delta\left(x-a\right)$$

The field momentum time derivative is due to the singular $\partial B/\partial t$ which is also discussed in Eq. (1.1)

$$\varepsilon_0\mu_0\frac{\partial\vec{S}_x}{\partial t} = \varepsilon_0\frac{\partial\left(\vec{E}\times\vec{B}\right)_x}{\partial t} = -\varepsilon_0\left[\frac{J_0}{\sigma}\right]_z\frac{\partial\vec{B}_y}{\partial t}\xrightarrow{y=\phi} -\varepsilon_0\left[\frac{J_0}{\sigma}\right]\left[-\frac{J_0}{\sigma}\delta\left(x-a\right)\right]_{\frac{\partial\vec{B}_y}{\partial t}} = \frac{\varepsilon_0 J_0^2}{\sigma^2}\delta\left(x-a\right)$$

Thus the singular $\vec{f}_x$ terms cancel:

$$\vec{f}_x^{\text{singular}} = \frac{\partial T_{xx}^{\text{elect}}}{\partial x} - \varepsilon_0\mu_0\frac{\partial\vec{S}_x}{\partial t} = \frac{\varepsilon_0 J_0^2}{\sigma^2}\delta\left(x-a\right) - \frac{\varepsilon_0 J_0^2}{\sigma^2}\delta\left(x-a\right) = 0 \to \mathcal{P}_{\text{outward}} = 0$$

We thus confirm that there is no pressure on the cylindrical surface of the resistor.