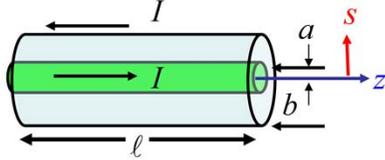
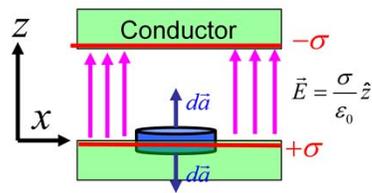


Conservation Laws

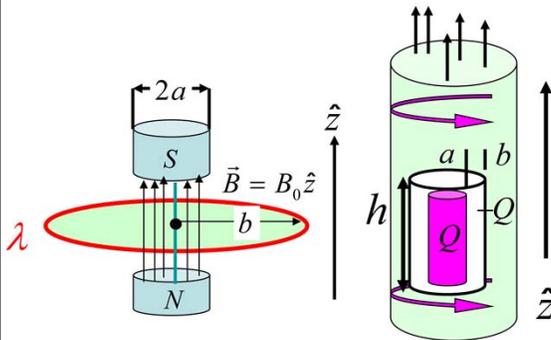
Poynting's Theorem



Maxwell Stress Tensor



Field linear and angular momentum



$$\vec{T}(z > 0) = \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} -1 & & \\ & -1 & \\ & & +1 \end{pmatrix}$$

1

This chapter deals with conservation of energy, momentum and angular momentum in electromagnetic systems. The basic idea is to use Maxwell's Eqn. to write the charge and currents entirely in terms of the E and B-fields. For example, the current density can be written in terms of the curl of B and the Maxwell Displacement current or the rate of change of the E-field. We could then write the power density which is E dot J entirely in terms of fields and their time derivatives. We begin with a discussion of Poynting's Theorem which describes the flow of power out of an electromagnetic system using this approach. We turn next to a discussion of the Maxwell stress tensor which is an elegant way of computing electromagnetic forces. For example, we write the charge density which is part of the electrostatic force density (rho times E) in terms of the divergence of the E-field. The magnetic forces involve current densities which can be written as the fields as just described to complete the electromagnetic force description. Since the force is the rate of change of momentum, the Maxwell stress tensor naturally leads to a discussion of electromagnetic momentum density which is similar in spirit to our previous discussion of electromagnetic energy density. In particular, we find that electromagnetic fields contain an angular momentum which accounts for the angular momentum achieved by charge distributions due to the EMF from collapsing magnetic fields according to Faraday's law. This clears up a mystery from Physics 435. We will frequently re-visit this chapter since it develops many of our crucial tools we need in electrodynamics.

Poynting's Theorem

$$\begin{aligned}
 d\text{Power} &= d\vec{F} \cdot \vec{v} = d \frac{dW}{dt} = dq(\vec{v} \times \vec{B} + \vec{E}) \cdot \vec{v} \\
 &= \vec{E} \cdot \vec{v} \rho d\tau = \vec{E} \cdot \vec{J} d\tau \rightarrow \frac{\partial W}{\partial t} = \int_V \vec{E} \cdot \vec{J} d\tau \\
 \vec{J} &= \vec{\nabla} \times \frac{\vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \rightarrow \vec{E} \cdot \vec{J} = \frac{\vec{E} \cdot \vec{\nabla} \times \vec{B}}{\mu_0} - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \\
 \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\epsilon_0 \vec{E} \cdot \vec{E}}{2} \right) = \frac{\partial u_e}{\partial t} \text{ where } U_e = \int_V u_e d\tau \\
 \rightarrow \vec{E} \cdot \vec{J} &= \frac{\vec{E} \cdot \vec{\nabla} \times \vec{B}}{\mu_0} - \frac{\partial u_e}{\partial t} \\
 \vec{\nabla} \cdot (\vec{E} \times \vec{B}) &= \vec{B} \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot \vec{\nabla} \times \vec{B} = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{\nabla} \times \vec{B} \\
 \vec{\nabla} \cdot (\vec{E} \times \vec{B}) &= -\frac{\partial}{\partial t} \frac{\vec{B} \cdot \vec{B}}{2} - \vec{E} \cdot \vec{\nabla} \times \vec{B} \\
 \rightarrow \frac{\vec{E} \cdot \vec{\nabla} \times \vec{B}}{\mu_0} &= -\frac{\partial}{\partial t} \frac{\vec{B} \cdot \vec{B}}{2} - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \\
 \frac{\vec{E} \cdot \vec{\nabla} \times \vec{B}}{\mu_0} &= -\frac{\partial u_m}{\partial t} - \vec{\nabla} \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right) \\
 \rightarrow \vec{E} \cdot \vec{J} &= -\frac{\partial}{\partial t} (u_e + u_m) - \vec{\nabla} \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right)
 \end{aligned}$$

Let $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$ (Poynting vector)

Since $\frac{dW}{dt} = \int_V \vec{E} \cdot \vec{J} d\tau$ we can think of

$\vec{E} \cdot \vec{J} = \frac{\partial u_{mech}}{\partial t}$ and write last expression as

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial}{\partial t} (u_e + u_m + u_{mech}) = 0$$

which resembles $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

\vec{S} is a energy density current.

The integral form would be:

$$\frac{dW}{dt} = \int_V u_{mech} d\tau = -\frac{\partial U_e + U_m}{\partial t} - \int_S \vec{S} \cdot d\vec{a}$$

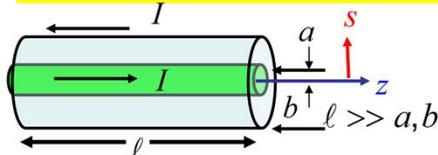
$$\text{where } \int_S \vec{S} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{S} d\tau$$

2

We begin with energy conservation and obtain the Poynting vector which you have had exposure to in Physics 212. Poynting was Maxwell's graduate student. The Poynting vector \vec{S} is very similar to the current density \vec{J} . Just like the current density gives the current per unit area flowing into a region, the Poynting vector gives the electromagnetic intensity or the power per unit area flowing into a region. We start with an element of power which is force dotted into velocity. Later we will call this the "mechanical" power density which is essentially the rate of change of the work done on the charges by the electromagnetic field per unit volume. As you will see, besides this mechanical energy there is also the energy density stored in the electric and magnetic fields themselves. For the "mechanical" power, the force is just the electric and magnetic force times for an element of charge dq dotted into velocity where only the E-field counts since the B-field does no work. The element of charge is ρ times an element of volume. \vec{J} is just $\rho \vec{v}$ and hence the total work done on the charge by the field per unit volume is $\vec{E} \cdot \vec{J}$. The basic strategy for this chapter is to use Maxwell's Eqn. to eliminate all sources in favor of field expressions. In this case, we can eliminate \vec{J} from the Maxwell-Ampere law which gives us one term from Ampere's law and one term from Maxwell's displacement current. The displacement current power contribution is then proportional to $\vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$, and hence Maxwell term can be written as the time derivative of $\epsilon_0 \vec{E} \cdot \vec{E} / 2$ which you recognize as the electric field energy density which we will call u_e . The volume of u_e is the total energy stored in the electric field as we discussed in the Potential Chp. The Ampere law contribution which is the B field dotted into the curl of E can be re-written using one of the product rules in Griffiths' Chp. 1. After this product rule is applied one is left with the time derivative of the magnetic energy u_m and the divergence of $\vec{E} \times \vec{B} / \mu_0$. The quantity $\vec{E} \times \vec{B} / \mu_0$ is the Poynting vector \vec{S} .

We can write $\vec{E} \cdot \vec{J}$ as the power density or rate of change of a mechanical energy density u_{mech} . The rate of change of the integral of u_{mech} is the total power flowing into the volume. With the u_{mech} definition, we can write our power expression as the divergence of \vec{S} plus the rate of change of the sum of all three densities (mechanical, electrical and magnetic) is zero. This is in close analogy with the continuity Eq. that says the divergence of \vec{J} gives the rate of change of the charge density (free and bound). Just like the surface integral of \vec{J} is the current flow into or out of a volume, the surface integral of \vec{S} is the power flowing into or out of a volume.

Poynting vector and the coaxial cable



Lets compute power going into cable via Poynting's Thm. Begin by constructing \vec{S}

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}; \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \text{ from Gauss's Law}$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{I}{2\pi s} \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \times \hat{\phi} = \frac{\lambda I}{\epsilon_0 (2\pi)^2 s^2} \hat{z}$$

Since we are in steady state $\frac{\partial(U_e + U_m)}{\partial t} = 0$

Compute power flowing into coaxial cable

$$\text{Power} = -\int_s \vec{S} \cdot d\vec{a}; d\vec{a} = -2\pi s \hat{z}$$

$$\text{Power} = \int_a^b \frac{\lambda I}{\epsilon_0 (2\pi)^2 s^2} 2\pi s ds = \frac{\lambda I}{2\pi\epsilon_0} \ln \frac{b}{a}$$

Same power flows out so $\int_s \vec{S} \cdot d\vec{a} = 0$

when both ends are considered.

We can write this in terms of voltage using

$$V = \int_b^a \vec{E} \cdot d\vec{\ell} = \int_b^a \frac{\lambda}{2\pi\epsilon_0 s} ds = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a}$$

$$\rightarrow \text{power} = \frac{dW}{dt} = VI$$

Check the divergence.

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u_{EM}}{\partial t} + \vec{E} \cdot \vec{J} = 0.$$

$$\frac{\partial u_{EM}}{\partial t} = 0 \text{ since } \vec{E} \text{ \& } \vec{B} \text{ are constant.}$$

$$\vec{E} \cdot \vec{J} \rightarrow \vec{E} \cdot \vec{K} = \frac{\lambda \hat{s}}{2\pi\epsilon_0 s} \cdot |K_{a,b}| \hat{z} = 0$$

Thus $\vec{\nabla} \cdot \vec{S}$ should vanish and it does

$$\vec{\nabla} \cdot \vec{S} = \frac{\partial S_z}{\partial z} = \frac{\partial}{\partial z} \left\{ \frac{\lambda I}{\epsilon_0 (2\pi)^2 s^2} \right\} = 0$$

3

Our coaxial cable where current flows down the inner conductor and back along the outer conductor gives an example of the use of the Poynting Vector. To compute \vec{S} we need an expression for \vec{B} and \vec{E} . We obtained \vec{B} using Ampere's law in our treatment of Griffiths Ex 7.13. This \vec{B} field is confined to $a < s < b$. To compute \vec{E} we need to know something about the charges. A simple assumption is the inner conductor has a linear charge density of λ and the outer conductor has a charge density of $-\lambda$. This is indeed what nature would set up if we put a voltage V between the two conductors. The current I could then be viewed as $2\pi a \lambda v$ in the same way we defined currents in our Magnetostatics Chp. But since we don't know the charge velocity we treat λ and I as independent variables. The same current is returned on the outer conductor so \vec{B} disappears at $s > b$. We can then construct \vec{E} by applying Gauss's Law using a Gaussian cylinder. The Poynting vector is just the cross product of \vec{E} and \vec{B} which points in the \hat{z} direction or the direction of the inner conductor current. Integrating \vec{S} over the area of the leftmost cylinder end gives the total power flowing into the coaxial cable. We are using the Gauss's law "outward" area convention which accounts for the $-$ sign. We get a simple expression involving the current, charge density and the $\log(b/a)$. This power should just be the usual VI power. To check this we need to compute the voltage by integrating our \vec{E} -expression along s from "a" to "b". This gives us a (hopefully) familiar expression involving λ and $\log(b/a)$. Indeed the product of V and I is the same as the surface integral of the Poynting vector \vec{S} . Although the power flows into the coax, it also flows out through the other end so the net power flow is zero and the energy inside the cable is constant. The situation is analogous to current flow through a wire. The same current flows into the wire that leaves the wire so the charge within the wire is constant. Since the total surface integral of the Poynting vector vanishes, the divergence theorem tells us that the volume integral of the divergence of \vec{S} must vanish. In fact the divergence of \vec{S} is zero for this case since the \vec{E} and \vec{B} fields are constant meaning their energy densities are constant and \vec{E} is perpendicular to \vec{J} so $\vec{E} \cdot \vec{J}$ vanishes. Indeed the divergence of our \vec{S} expression is zero.

Poynting vector and the cylindrical resistor

$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$; $\vec{J} = \sigma \vec{E} \rightarrow \vec{E} = \frac{J_0}{\sigma} \hat{z}$
 $\int \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} \rightarrow 2\pi s B_\phi = \mu_0 \pi s^2 J_0$
 $\vec{B} = \frac{\mu_0 s J_0 \hat{\phi}}{2} \rightarrow \vec{S} = \frac{1}{\mu_0 \sigma} \hat{z} \times \frac{\mu_0 s J_0 \hat{\phi}}{2} = -\frac{s J_0^2 \hat{s}}{2\sigma}$
 Check $\vec{\nabla} \cdot \vec{S} + \frac{\partial u_{EM}}{\partial t} + \vec{E} \cdot \vec{J} = 0$; $\frac{\partial u_{EM}}{\partial t} = 0$
 $\vec{\nabla} \cdot \vec{S} = \frac{1}{s} \frac{\partial}{\partial s} \left\{ -s \frac{s J_0^2}{2\sigma} \right\} = -\frac{J_0^2}{\sigma}$
 $\vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \vec{J} = \left[-\frac{J_0^2}{\sigma} \right]_{\vec{\nabla} \cdot \vec{S}} + \left[\frac{J_0}{\sigma} \hat{z} \right]_{\vec{E}} \cdot [J_0 \hat{z}]_J = 0$

$$\int_S \vec{S} \cdot d\vec{a} = \int_{\#1} \vec{S} \cdot d\vec{a}_{\#1} + \int_{\#2} \vec{S} \cdot d\vec{a}_{\#2} + \int_{\#3} \vec{S} \cdot d\vec{a}_{\#3}$$

$$\vec{S} \cdot d\vec{a}_{\#1} = -\frac{s J_0^2 \hat{s}}{2\sigma} \cdot (-2\pi s ds \hat{z}) = 0 \quad \& \quad \vec{S} \cdot d\vec{a}_{\#2} = 0$$

$$\vec{S} \cdot d\vec{a}_{\#3} = -\frac{a J_0^2 \hat{s}}{2\sigma} \cdot (+ahd\phi \hat{s})$$

$$\rightarrow \int_S \vec{S} \cdot d\vec{a} = -\frac{\pi a^2 h J_0^2}{\sigma} \text{ power flows into resistor}$$

Lets check against familiar power expression

$$\text{Power} = \Delta VI = \left(\frac{J_0 h}{\sigma} \right)_{\Delta V} (\pi a^2 J_0)_I = \frac{\pi a^2 h J_0^2}{\sigma} = \int_S \vec{S} \cdot d\vec{a}$$

Power \rightarrow mech $\frac{\partial U_{mech}}{\partial t} = \int \vec{E} \cdot \vec{J} d\tau \rightarrow \text{heat} + \frac{\partial (KE + PE)}{\partial t}$

power = $\frac{\partial U_{EM}}{\partial t} + \int \vec{E} \cdot \vec{J} d\tau = \left(\frac{J_0 \hat{z}}{\sigma} \right)_E \cdot (J_0 \hat{z})_J (\pi a^2 h)_\tau$

But $\vec{S}(s > a) = \frac{\vec{E}(s > a) \times \vec{B}}{\mu_0} = 0$

$\rightarrow \left[\int \vec{S} \cdot d\vec{a} \right]_{cylinder}^{R>a} = 0 \neq \frac{\pi a^2 h J_0^2}{\sigma}$ **whoops!**

Any theories?

4

Lets apply the Poynting vector approach to a cylindrical resistor of radius a , length h and conductivity σ . Recall in a resistor, the E-field is proportional to J and J is a constant vec $J = J_0 \hat{z}$. We can find the B-field within the resistor using Ampere's law and we find that it is proportional to s and points in the $\hat{\phi}$ direction. We then form the cross product of E and B to get a Poynting vector which points in the $-\hat{s}$ direction. This is a bit surprising since one would naively think the power flows into the ends rather than into the resistor through the cylinder walls. We next compute the total power flow out of the resistor by integrating the Poynting vector. We need to consider the two ends (#1 and #2) and the barrel (#3). Since the end area vectors are in the $\pm \hat{z}$ direction and the Poynting vector is in the $-\hat{s}$ direction, only the barrel region contributes. We get a simple integral over $d\phi$ for the barrel and a negative power flow out of the resistor indicating that the power flows into the resistor. We all know that the power into a resistor is VI and we can calculate V from $V = hE$. The current is just $J_0 \pi a^2$. We can insert I and E into our power expression to get exactly the same power that we computed using the Poynting vector. But where does this power go? The electric and magnetic fields are both static so it cannot go into changing U_{EM} . The power **does** go into mechanical energy since it equals $E \cdot J d\tau$. The current and voltage are constant so it cannot be increasing the $KE + PE$ of the charges. Evidently the mechanical power goes into heat. This heat presumably radiates radially from the resistor, so perhaps it makes sense that the Poynting vector flows radially into the resistor to compensate for the heat loss. But something doesn't make sense when we try to compute the power with a cylinder with a radius larger than " a ". The problem is that the electrical field disappears outside of the resistor material which implies that $S(s > a) = 0$. Hence we seem to be violating Poynting's Theorem which says that $\int S \cdot \text{vec} \cdot d\vec{a}$ is the power contained within the boundary. Poynting's Theorem works fine if our bounding volume just contains the resistor, but if it becomes larger than the resistor we get **zero** power. Presumably there is a problem with Poynting's theorem, the divergence theorem, or our resistor model. Hint– does our resistor model satisfy Maxwell's equations?

Maxwell Stress Tensor

We start with the force on a volume of charge: $\vec{F} = \int_V \rho \vec{E} + \vec{J} \times \vec{B} \, d\tau$

Our goal is to use Maxwell's Eq. to replace ρ and \vec{J} with field expressions.

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}; \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \rightarrow \vec{J} = \frac{1}{\mu_0} \left(\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\text{Writing } \vec{F} = \int_V \vec{f} \, d\tau \rightarrow \vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} = \epsilon_0 \vec{E} \vec{\nabla} \cdot \vec{E} + \frac{1}{\mu_0} \left(\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

Our force density \vec{f} is indeed entirely written in terms of fields but does not have expected symmetry (e.g.) we have a $\vec{E} \vec{\nabla} \cdot \vec{E}$ term but no $\vec{B} \vec{\nabla} \cdot \vec{B}$ term

"Symmetrize" by writing $\frac{\partial \vec{E}}{\partial t} \times \vec{B}$ in terms of \vec{S} and use Faraday's law

$$\frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t} = \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \vec{E} \times (\vec{\nabla} \times \vec{E})$$

$$\rightarrow \boxed{\frac{\partial \vec{E}}{\partial t} \times \vec{B} = \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\vec{\nabla} \times \vec{E})}$$

5

Our goal is to write the electromagnetic pressure on volume of charge entirely in terms of electromagnetic fields. We begin by writing the force in terms of volumetric densities both charge and current. Our approach is to replace the charge and current with field quantities. The charge density is simple since it is essentially the divergence of the E-field. The current density is slightly more complicated since it involves the curl of the B-fields and the rate of change of the E-field (essentially the Maxwell displacement current density). Since we have so many volume integrals we switch to the force density rather than the force. We are able to fairly rapidly get to a pure field expression, but our expression doesn't have the basic E and B symmetry we've learned to expect in electrodynamics. One major "symmetry-breaker" is the term involving the curl of B crossed into B – there is clearly no such term in E. But we can manipulate the other "symmetry-breaker" involving the time derivative of E crossed into B by using the derivative of a product – and our product is essentially the Poynting vector. One term is our desired time derivative of E crossed into B and the other is E crossed into the time derivative of B. But this can be converted into the curl of E crossed into E using Faraday's law which will restore symmetry with the curl of B crossed into B term.

The Stress continues...

Hence we write $\vec{f} = \epsilon_0 \vec{E} \vec{\nabla} \cdot \vec{E} + \frac{1}{\mu_0} \left((\vec{\nabla} \times \vec{B}) \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right)$ as

$$\vec{f} = \epsilon_0 \vec{E} \vec{\nabla} \cdot \vec{E} + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} - \epsilon_0 \left(\frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\vec{\nabla} \times \vec{E}) \right)$$

$$\vec{f} = \epsilon_0 \vec{E} \vec{\nabla} \cdot \vec{E} - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \times (\vec{\nabla} \times \vec{E}) - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

Its looking fairly $E \leftrightarrow B$ symmetric apart from $\epsilon_0 \vec{E} \vec{\nabla} \cdot \vec{E}$ term

This is easy to fix since $\vec{\nabla} \cdot \vec{B} = 0$ so we can add $\vec{B} \vec{\nabla} \cdot \vec{B} / \mu_0$ to get

$$\vec{f} = \epsilon_0 \left[\vec{E} \vec{\nabla} \cdot \vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E}) \right] + \frac{1}{\mu_0} \left[\vec{B} \vec{\nabla} \cdot \vec{B} - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

The last term $\epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \epsilon_0 \mu_0 \frac{\partial}{\partial t} \vec{S}$ is essentially rate of change of Poynting's Vector but what are the the first two terms?

6

Using this Faraday's law trick we are able to get a nearly perfectly symmetric form between B and E (and epsilon_0 and 1/mu_0). The only symmetry breaker is E times the divergence of E. But this symmetry is easy to restore since the divergence of B is zero which means we can simply add B times the divergence of B with the appropriate epsilon_0 to mu_0 substitution to get a "perfect" form for the force density involving the fields and the time derivative of the Poynting vector. The only problem is that there are lots of complicated terms.

Introducing the Tensor

After some tedious algebra we can write these terms in "component form":

$$\vec{E} \cdot \vec{\nabla} - \vec{E} \times (\vec{\nabla} \times \vec{E}) = \sum_{i=x,y,z} \frac{\partial}{\partial x_i} \left(E_i E_j - \frac{\delta_{ij}}{2} \vec{E} \cdot \vec{E} \right) \text{ where } \delta_{ij} = 1 \text{ if } i = j \text{ and } 0 \text{ otherwise}$$

We have a similar looking term for the B field.

$$\vec{f}_j = \sum_{i=x,y,z} \frac{\partial}{\partial x_i} \left({}^{(\text{elect})}T_{ij} + {}^{(\text{mag})}T_{ij} \right) - \epsilon_0 \mu_0 \frac{\partial S_j}{\partial t}$$

Lets use the notation ${}^{(\text{elect})}T_{ij} = \epsilon_0 \left(E_i E_j - \frac{\delta_{ij}}{2} \vec{E} \cdot \vec{E} \right)$ and ${}^{(\text{mag})}T_{ij} = \frac{1}{\mu_0} \left(B_i B_j - \frac{\delta_{ij}}{2} \vec{B} \cdot \vec{B} \right)$

The two T_{ij} forms are second rank tensors and can be written as matrices

$${}^{(\text{elect})}T = \frac{\epsilon_0}{2} \begin{pmatrix} E_x E_x - E_y E_y - E_z E_z & 2E_x E_y & 2E_x E_z \\ 2E_x E_y & E_y E_y - E_x E_x - E_z E_z & 2E_y E_z \\ 2E_x E_z & 2E_y E_z & E_z E_z - E_x E_x - E_y E_y \end{pmatrix}$$

See if you can get this matrix from Kronecker delta expression!

7

Griffiths shows how the E-field and B-field vector expressions can be algebraically manipulated into the simpler, more memorable form of the Maxwell stress tensor. The original form is a vector which is bilinear (eg two powers) in the E-field and one derivative. The final form is still bilinear in the fields and involves derivatives but is not of the usual vector calculus form involving curls and divergences. Hence we write our bilinear expression in component form where the sum over i is a sum over the 1,2,3 or x,y,z component. The delta_ij is a convenient shorthand called the Kronecker delta which is zero unless i = j and is 1 otherwise. You can think of it as sort of a discrete version of the Dirac delta function delta (r - r') which we introduced the Dirac delta function way back in Physics 435 to show the equivalence of Coulomb's law and Gauss's law. The original bilinear form in the E-fields (EiEj - E dot E delta_ij/2) has two indices and holds 9 components. A vector - on the other hand has 3 indices i = 1,2,3 for x,y,z. This new object (EiEj - E dot E delta_ij/2) is not a scalar (one component) or a vector (3 components) and is given a new name (a second rank) tensor which has 9 components. Your text writes it with a double headed arrow to emphasis the fact our bilinear form is a tensor and not a scalar or vector. The 9 components are most conveniently organized as a table with three rows and three columns which is the same as a three by three matrix. In our force density expression one takes the derivative with respect to x_i. After we sum over the i index when computing the derivative this leaves us one index (j) and hence the force expression contains one index j where j denotes the force component. The tensor T_ij components for the electric field part can be displayed as the indicated 3 by 3 matrix. We note that the diagonal components involve all three components and the off diagonal components involve just two.

The divergence of a tensor

We can write ${}^{(\text{elect})}\vec{f}_j = \sum_{i=x,y,z} \frac{\partial}{\partial x_i} {}^{(\text{elect})}\vec{T}_{ij}$ as pre-multiplication of the ${}^{(\text{elect})}\vec{T}$ the tensor

by the vector $\vec{\nabla} = \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right)$ or ${}^{(\text{elect})}\vec{f}_j = \sum_{i=x,y,z} \vec{\nabla}_i {}^{(\text{elect})}\vec{T}_{ij}$. Can be written as

matrix multiplication since the adjacent indices summed. $A_i = \sum_j B_{ij} C_j$ or $\underline{A} = \underline{BC}$

$$\left({}^{(\text{el})}\vec{f}_x \quad {}^{(\text{el})}\vec{f}_y \quad {}^{(\text{el})}\vec{f}_z \right) = \frac{\epsilon_0}{2} \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} E_x E_x - E_y E_y - E_z E_z & 2E_x E_y & 2E_x E_z \\ 2E_x E_y & E_y E_y - E_x E_x - E_z E_z & 2E_y E_z \\ 2E_x E_z & 2E_y E_z & E_z E_z - E_x E_x - E_y E_y \end{pmatrix}$$

We multiply out this matrix to get the various components for the force. For example:

$${}^{(\text{elect})}\vec{f}_x = \frac{\epsilon_0}{2} \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} E_x E_x - E_y E_y - E_z E_z \\ 2E_x E_y \\ 2E_x E_z \end{pmatrix} = \frac{\epsilon_0}{2} \left[\frac{\partial}{\partial x} (E_x E_x - E_y E_y - E_z E_z) + \frac{\partial}{\partial y} (2E_x E_y) + \frac{\partial}{\partial z} (2E_x E_z) \right]$$

A natural notation for ${}^{(\text{elect})}\vec{f}_j = \sum_{i=x,y,z} \frac{\partial}{\partial x_i} {}^{(\text{elect})}\vec{T}_{ij}$ is thus ${}^{(\text{elect})}\vec{f} = \vec{\nabla} \bullet {}^{(\text{elect})}\vec{T}$ which means

pre-multiply the 3 by 3 ${}^{(\text{elect})}\vec{T}$ matrix by the $\vec{\nabla}$ vector. We can think of $\vec{\nabla} \bullet \vec{T}$ as the divergence of the tensor \vec{T} which returns a vector. This is in contrast to the divergence of a vector (such as $\vec{\nabla} \bullet \vec{E}$) which returns a scalar ($\epsilon_0 \rho$).

The full expression for the force density can then be written as:

$$\vec{f} = \vec{\nabla} \bullet \vec{T} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} \quad \text{where the components are } \vec{T}_{ij} = \epsilon_0 \left(E_i E_j - \frac{\delta_{ij}}{2} \vec{E} \bullet \vec{E} \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{\delta_{ij}}{2} \vec{B} \bullet \vec{B} \right) \quad 8$$

We begin by writing the electric force piece f_i due to the E-field for the static case (so we can ignore the Poynting vector piece) in component form as a sum over the i component. We can write the f_i in the form of the multiplication

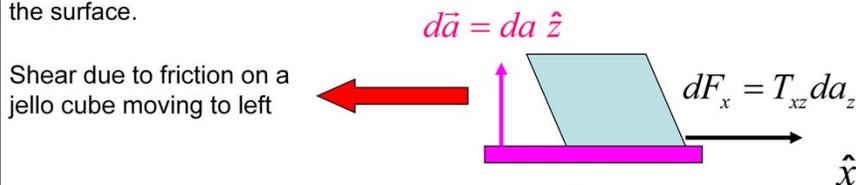
of the del vector by the stress tensor matrix T. We can view it as the multiplication of a 1 by 3 matrix (the del row vector) times a 3 by 3 matrix (the stress tensor) to give a 1 by 3 matrix (the f row vector). The trick to recognizing the matrix multiplication is that adjacent, repeated indices are summed (in this case the j – index is the adjacent, repeated index). We give you an example of multiplication of matrix B times C to give matrix A. We also show the explicit (table) multiplication of the del vector times the electrical part of the field tensor to give electrical part of all three components of the electric force. We can think of the f row vector as consisting of three divergences if we think of T as being constructed from three column vectors. We show how the x-component of f is constructed from the divergence of the first column vector piece of T. I think of this as taking the “dot product” of the del vector with the first column vector of T. We conclude by showing all three pieces of the force expression, the electrical piece, magnetic piece and the Poynting vector derivative.

The stress tensor

We can apply the divergence theorem ($\int_{Vol} \vec{\nabla} \cdot \vec{V} d\tau = \int_S \vec{V} \cdot d\vec{a}$) to the tensor divergence ($\vec{\nabla} \cdot \vec{T}$)

to write $\vec{F} = \int_V \vec{f} d\tau = \int_S \vec{T} \cdot d\vec{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} d\tau$ where \vec{F} is force on all charges within V .

We have gone to a total time derivative over $\int_V \vec{S} d\tau$ since after integration there are no other dependences apart from time. Note \vec{T} has the dimensions of a pressure or force per unit area. But unlike pressure \vec{T} can describe a shear force which is not \perp to the surface.



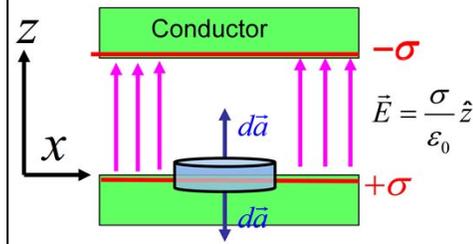
$$\text{For } \partial \vec{S} / \partial t = 0 : \vec{F}_x(q, \vec{J} \subseteq \mathcal{S}) = \int_S \vec{T}_{xx} da_x + \vec{T}_{xy} da_y + \vec{T}_{xz} da_z$$

The diagonal components of T are pressures (forces normal to the surfaces).
The off diagonal components of T are shears (forces parallel to surfaces). ⁹

The tensor divergence is constructed from three actual divergences of three vector fields. As we saw in the previous slide, for example, the f_x piece of the tensor divergence shown on the previous slide is the divergence of the first column vector of the T -matrix. We can thus apply the divergence theorem to each separate column vector of the tensor divergence and replace the volume integral of the divergence as a surface integral over the surface that bounds the charge volume. Unlike the surface integral of say the vector E dot $d\vec{a}$ in Gauss's law where we get a scalar (the enclosed charge), we will get three components for the tensor

$T \cdot d\vec{a}$ which for static fields (where the Poynting vector is constant) will be the three components of the force acting on the charge within the volume. This form where a tensor is dotted into an area to give a force represents a generalization of the concept of a pressure which often shows up in the mechanics of solids as the stress tensor –developed by Cauchy. In general a solid can respond to both pressures (forces perpendicular to the surface) and shears (forces parallel to the surface). The picture illustrates the deformation of a jello cube sliding on a rough surface. The friction creates a force on the contact surface that opposes the motion creating a shear force. The Maxwell stress tensor describes pressures (diagonal elements) as well as shears (the off diagonal elements). Amazingly it is able to describe these shears and pressures on a volume of charges entirely through the fields on the surface.

Electrostatic example



$$\vec{S} = 0, \vec{B} = 0 \rightarrow \vec{F} = \int_S \vec{T} \cdot d\vec{a}$$

$$\text{where } \vec{T}_{ij} = \epsilon_0 \left(E_i E_j - \frac{\delta_{ij}}{2} \vec{E} \cdot \vec{E} \right)$$

$$\text{For } z > 0, \vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} \text{ and for } z < 0, \vec{E} = 0$$

$$\text{Since } E_x = E_y = 0, \vec{T}_{ij} = 0 \text{ unless } i = j$$

Only have pressures and no shear

$$\vec{T}_{xx} = \vec{T}_{yy} = \epsilon_0 \left(-\frac{1}{2} E_z E_z \right) = -\frac{\sigma^2}{2\epsilon_0}; \vec{T}_{zz} = \epsilon_0 \left(E_z E_z - \frac{1}{2} E_z E_z \right) = +\frac{\sigma^2}{2\epsilon_0}$$

$$\Rightarrow \vec{T}(z > 0) = \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} -1 & & \\ & -1 & \\ & & +1 \end{pmatrix} \text{ for squat cylinder only } d\vec{a} = \pm da \hat{z} \text{ exists and only}$$

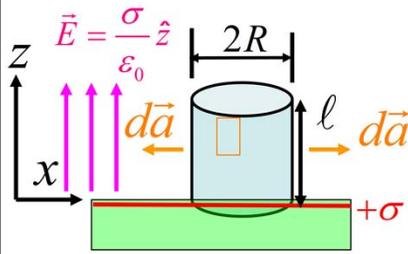
$$\text{upper surface has } \vec{T} \neq 0 \text{ Hence } \vec{F} = \int_S \vec{T} \cdot d\vec{a} \rightarrow \vec{F}_i = \sum_{j=x,y,z} \int_S \vec{T}_{ij} d\vec{a}_j; d\vec{a} = da(0 \ 0 \ 1)$$

$$\text{Hence } \vec{F}_i = \int_S \vec{T}_{iz} da \text{ and } \vec{T}_{iz} \neq 0 \text{ only for } \vec{T}_{zz} \Rightarrow \vec{F} = \hat{z} \int_S \vec{T}_{zz} da = \frac{\sigma^2}{2\epsilon_0} a \rightarrow \frac{\vec{F}}{\text{area}} = \frac{\sigma^2}{2\epsilon_0} \hat{z}$$

which is the familiar answer from way back in Physics 435

Here we compute forces on the lower charges in an ideal capacitor using the stress tensor. I do this example to relieve **your** stress at using the stress tensor formalism. The nice thing about this example is that the field is only in the z-direction. The matrix form on slide 10 shows the off-diagonal components require two non-vanishing E-field components so our stress tensor is diagonal thus no shears exist – only pressures. Although the E-field is only in the z-direction, our component rule shows all three diagonal pressures exist Txx, Tyy, and Tzz. No magnetic fields exist, so the Poynting vector is zero meaning its time derivative vanishes. To find the force on some charges, we need to find the surface integral over T over a surface that bounds the charges. We know that for an ideal conductor all the charges reside on the top surface of the lower conductor. A simple surface that contains all of the charges within an area “a” would be a squat, cylindrical pill box that extends just above and just below the surface. For an infinitesimal pill box there cylinder area vanishes and all of the area is on the top (pointing along z-hat) and the bottom (pointing along -z-hat). There is no E-field within the conductor thus T vanishes on the bottom surface leaving only the top surface which contributes a force vector of (Txx Area_z, Tyy Area_z, Tzz Area_z). The only non-vanishing component is Tzz which means the force within the pill box is entirely in the z-direction. Inserting our Tzz answer we get a force/unit area given by sigma^2/(2 epsilon_0). This is 1/2 of E times sigma since as we saw in Physics 435, only 1/2 of the E field is due to charges other than the charges in the patch of charge we are considering.

What if we use a longer cylinder to contain lower charge?



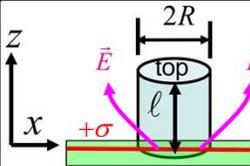
We should get same F_z answer but for a longer cylinder we might have non-vanishing F_x or F_y from "cylinder" part of surface $\vec{F} = \int_S \vec{T} \cdot d\vec{a}$ where $d\vec{a} = \hat{s} R l d\phi$; Lets start with $F_x = \int_S \hat{x} \cdot \vec{T} \cdot d\vec{a}$

Since $\vec{T} = \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} -1 & & \\ & -1 & \\ & & +1 \end{pmatrix}$; $\hat{x} \cdot \vec{T}$ is due to T_{xx}

$$\vec{F} = \int_S \vec{T} \cdot d\vec{a} \text{ where } d\vec{a} = \hat{x} R l d\phi; F_x = \int_S \vec{T}_{xx} \cdot \hat{x} \cdot d\vec{a}$$

$$\rightarrow F_x = \int \vec{T}_{xx} \cdot \hat{x} \cdot \hat{s} R l d\phi = -\frac{\sigma^2}{2\epsilon_0} \int_0^{2\pi} \hat{x} \cdot (\cos\phi \hat{s}_x + \sin\phi \hat{s}_y) R l d\phi = -\frac{\sigma^2}{2\epsilon_0} \int_0^{2\pi} \cos\phi R l d\phi = 0$$

What if $E_z(z)$? Won't $F_z = \int_{\text{top}} T_{zz} da_z$ depend on arbitrary l ?



If $\frac{\partial E_z}{\partial z} \neq 0 \Rightarrow E_{x,y} \neq 0$ since $\vec{\nabla} \cdot \vec{E} = 0 \rightarrow T_{zx} \& T_{zy} \neq 0$

\therefore Shear $\int_{\text{cylinder}} T_{zx} da_x + T_{zy} da_y$ cancels $\frac{\partial}{\partial l} \left[\int_{\text{top}} T_{zz} da_z \right]$

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The stress tensor expression for Force due to some charge should be true for **any** surface that contains these charges and no other charge. Hence it should hold for the longer cylindrical pill box illustrated above. This is similar to Gauss's Law applied to a spherical ball of charge Q with radius R which says that the area integral of the electrical field equal to Q/ϵ_0 as long as the surface lies entirely outside of R . Lets first consider the case of infinite radius capacitor plates.

The argument for F_z based on the top surface will go through unchanged since E_z is independent of z . But now we have a non-vanishing cylindrical area and hence we might have shear contributions due to T_{zx} and T_{zy} in addition to the pressure contribution due to T_{zz} . Of course if the net shear is non-zero we will have an F_x or F_y contribution and our formalism leads to the wrong answer. Lets compute F_x . This will involve the x -component of the area vector $d\vec{a}$ and the constant T_{xx} tensor component. The area vector points in the \hat{s} direction which involves \hat{s}_x or $\cos(\phi)$. The area element is $R L d\phi$ where L is the length of our pill box. Hence F_x is proportional to the integral of $\cos(\phi) d\phi$ which vanishes. The same would apply to F_y . So the squat and tall cylinder both give the correct answer for the charge contained on the top surface of the lower conductor. What happens if the E_z is not constant but falls with increasing z ? The figure shows how this can happen by having the field lines diverge. But this appears to cause a contradiction. As the cylinder becomes longer, T_{zz} on the top surface decreases implying that the force on the contained surface charge depends on arbitrary cylinder length. Fortunately Gauss's law saves the day! If E_z depends on z , we must have a non-zero E_x and E_y to insure that the field divergence is zero. This is illustrated in our field line drawing. A non-zero E_x and E_y implies that there will be shear terms T_{zx} and T_{zy} in addition to T_{zz} . Presumably these shears act on the cylinder surface. As cylinder length increases, the total z -shear increases which evidently compensates for the falling pressure contribution of the top plane. A simple demonstration model is :

$$\text{vec } E = (E_0 - \beta z) \hat{z} + (\beta/2)[x \hat{x} + y \hat{y}] \text{ where } |\beta| \ll 1.$$

You can easily show this field has zero divergence. If you keep only linear β terms in the stress tensor you can demonstrate that all of the β terms vanish from the total force expression for any assumed cylinder height thus removing the contradiction(see long cylinder in Schedule).

When do we get surface pressure?

$E_z^T = \bar{E} + \frac{\sigma}{2\epsilon_0}$
 $E_z^B = \bar{E} - \frac{\sigma}{2\epsilon_0}$

$$F_z = \bar{E} \mathcal{A} \sigma; \bar{E} = \frac{E_z^T + E_z^B}{2}; \sigma = \epsilon_0 (E_z^T - E_z^B)$$

$$F_z = \bar{E} \mathcal{A} \sigma = \left[\frac{E_z^T + E_z^B}{2} \right] \mathcal{A} \left[\epsilon_0 (E_z^T - E_z^B) \right]$$

$$F_z = \mathcal{A} \frac{\epsilon_0}{2} \left[(E_z^T)^2 - (E_z^B)^2 \right] = \mathcal{A} T_{zz}^T - \mathcal{A} T_{zz}^B$$

$$\vec{f}_z = (\nabla \cdot \vec{T})_z - \epsilon_0 \mu_0 \frac{\partial \vec{S}_z}{\partial t} = \frac{\partial T_{zz}}{\partial z} - \epsilon_0 \mu_0 \frac{\partial \vec{S}_z}{\partial t}; F_z = \int_{\text{pill box}} \vec{f}_z d\tau = \mathcal{A} \lim_{\delta \rightarrow 0} \left\{ \int_{s-\delta}^{s+\delta} \left[\frac{\partial T_{zz}}{\partial z} - \epsilon_0 \mu_0 \frac{\partial \vec{S}_z}{\partial t} \right] dz \right\}$$

∴ Only $\frac{\partial T_{zz}}{\partial z}$ and $\epsilon_0 \mu_0 \frac{\partial \vec{S}_z}{\partial t}$ which are singular at S contribute to surface pressure.

$\frac{\partial T_{zz}}{\partial z} = \frac{\sigma}{\epsilon_0} \delta(z - S) \rightarrow \infty$

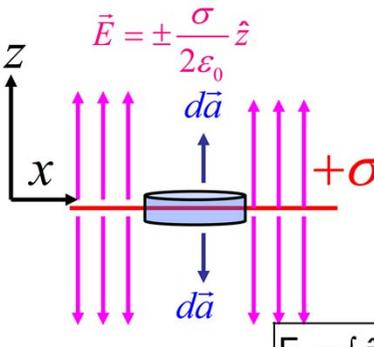
A surface charge σ or surface current \vec{K} gives a finite $\Delta \vec{T} \rightarrow$ singular $\nabla \cdot \vec{T}$

But one rarely has a singular $\epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$

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I would like to give you some more insight into the use of the Maxwell stress tensor to find the surface pressure. We get a surface pressure when there is a surface charge or surface current. In Physics 435, we computed the pressure on a conductor by multiplying the surface charge on a patch on the surface by E_{other} which is field due to all charges in a system **other** than the surface charge. In computing E_{other} , we model the patch as an infinite plane of charge which creates a field discontinuity by contributing $\pm \sigma/2\epsilon_0$ on either side of the patch. Our figure illustrates this superposition in the vicinity of patch which is normal to z -axis. Here $\bar{E} = E_{\text{other}}$ which as the algebra shows is the average of the field on infinitesimally close to but on either side of the patch. \bar{E} is due to all other charges in the system (which might be a cylindrical shell for example). The charge density σ is proportional to the difference in the E_z on either side of the patch. Multiplying the \bar{E} sum by the E difference gives the difference of the squared fields which is essentially the difference of T_{zz} on either side of the patch. We thus recover the pillbox Stress Tensor answer for the pressure. It is also useful to think of the pressure using the force density which is the divergence of the stress tensor minus the time derivative of the field momentum. To get a surface pressure we use an infinitesimal pill box which will require an infinite (singular) stress tensor divergence or dS/dt inside the pill box to produce a finite pressure. We can easily get a singular tensor divergence if the stress tensor is different on either side of the surface due to a surface charge or surface current. It is difficult to have a singular dS/dt at the pill box so this term can generally be neglected. I offer a counter example in my discussion of the **Poynting Paradox** on the "schedule" web page. You might want to test your skills by working out the **magnetic pressure** on a surface carrying a surface current. One can also easily extend this informal treatment to compute the surface shears as well as the pressure.

Another simple example: Single sheet of charge



Now $\vec{T} = \frac{\sigma^2}{8\epsilon_0} \begin{pmatrix} -1 & & \\ & -1 & \\ & & +1 \end{pmatrix}$ since $E = \frac{E_{\text{previous}}}{2}$

This is true for both $z > 0$ and $z < 0$ since \vec{T} is quadratic in \vec{E} . Here $d\vec{a} = \pm \hat{z} da$ which involves $\hat{z} \cdot \vec{T}$. Only nonzero $\hat{z} \cdot \vec{T}$ part is from $\vec{T}_{zz} = \frac{\sigma^2}{8\epsilon_0}$

$$\vec{F}_z = \int_S \hat{z} \cdot \vec{T} \cdot d\vec{a} = \int_{S(z>0)} \hat{z} \cdot \vec{T} \cdot \hat{z} da + \int_{S(z<0)} \hat{z} \cdot \vec{T} \cdot (-\hat{z} da)$$

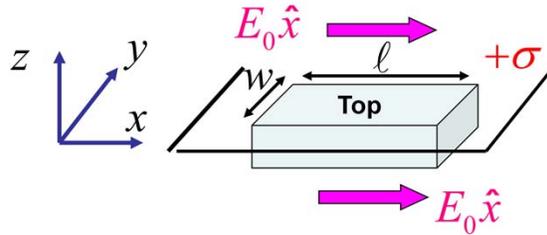
$$\vec{F}_z = \int [\vec{T}_{zz}(z > 0) - \vec{T}_{zz}(z < 0)] da = 0 \rightarrow \vec{F} = 0$$

This is indeed true since $\frac{\vec{F}}{a} = \sigma(\vec{E} - \vec{E}_{\text{self}})$

But $\vec{E} = \vec{E}_{\text{self}} = \pm \frac{\sigma}{2\epsilon_0} \hat{z} \rightarrow \vec{F} = 0$

Here is another simple case to help introduce the Maxwell stress tensor. We consider the force acting on an infinite single sheet of charge with a charge density σ . Gauss's law tells us the field is $\pm \sigma / (2 \epsilon_0) \hat{z}$. The resulting stress tensor is $1/4$ of what it was for the previous (two-sheet) case since the field is lower by a factor of two and the tensor goes as the square of the field. But now E extends both above and below the charge sheet. Although the E -field reverses as one passes through the charge plane, the field tensor does not since it is bilinear in the two fields. We use a pill-box surface that encloses an area of charge. The two area vectors point outward from the pill-box and are $\pm d \text{ area } \hat{z}$ as shown. Since the stress tensor is the same both above and below the plane, but the area vector changes sign, the force integral is zero and there is no electrostatic pressure on the charge. We would reach the same conclusion using Physics 435 methods since the pressure on a patch of charge would be the charge density times E_{other} where E_{other} is the total E -field minus the self field or the field due to the patch itself. Recall Gauss's law argument for the field due to a patch of surface charge concludes $E_{\text{self}} = \pm \sigma / (2 \epsilon_0)$ and hence the self-field equals the total field and hence there is no electrostatic pressure which is exactly what we conclude from the stress tensor integral. Incidentally it might seem paradoxical that there is no electrostatic pressure on a plane of charge since this would imply that no external force is required to overcome the Coulomb repulsion and hold the charge in a plane. I think it is apparent that the x and y force components must be zero since any force on a charge due to a charge on the left will be cancelled by the force due to a charge on the right. But what about the forces normal to the plane along the z -axis? But would the normal force be in the positive or negative direction? Clearly if a single charge were moved above the plane it would be forced upwards and if moved below the plane it would be forced downwards. I think in the plane it is in an unstable equilibrium and no force is required.

Shear Ecstasy



$$\vec{F} = \int_{\text{top}} \vec{T} \cdot d\vec{a} + \int_{\text{bottom}} \vec{T} \cdot d\vec{a} ; d\vec{a}^{\text{top}} = \hat{z} dx dy ; d\vec{a}^{\text{bottom}} = -\hat{z} dx dy$$

$$\begin{pmatrix} dF_x^{\text{top}} \\ dF_y^{\text{top}} \\ dF_z^{\text{top}} \end{pmatrix} = \frac{\epsilon_0}{2} \begin{pmatrix} E_x E_x - E_y E_y - E_z E_z & 2E_x E_y & 2E_x E_z \\ 2E_x E_y & E_y E_y - E_x E_x - E_z E_z & 2E_y E_z \\ 2E_x E_z & 2E_y E_z & E_z E_z - E_x E_x - E_y E_y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ dx dy \end{pmatrix}$$

$$dF_x^{\text{top}} = \frac{\epsilon_0}{2} 2E_x E_z dx dy \rightarrow F_x = \epsilon_0 \int_{-\ell/2}^{\ell/2} dx \int_{-w/2}^{w/2} dy E_x [E_z]^{\text{top}} - \epsilon_0 \int_{-\ell/2}^{\ell/2} dx \int_{-w/2}^{w/2} dy E_x [E_z]^{\text{bottom}}$$

$$\left[\vec{E} \right]_{\text{bottom}}^{\text{top}} = \begin{pmatrix} E_0 & 0 & \pm \frac{\sigma}{2\epsilon_0} \end{pmatrix} \rightarrow F_x = \epsilon_0 \int_{-\ell/2}^{\ell/2} dx \int_{-w/2}^{w/2} dy [E_0]_{E_x} \left[\frac{\sigma}{2\epsilon_0} \right]_{E_z} - \epsilon_0 \int_{-\ell/2}^{\ell/2} dx \int_{-w/2}^{w/2} dy E_0 \frac{-\sigma}{2\epsilon_0}$$

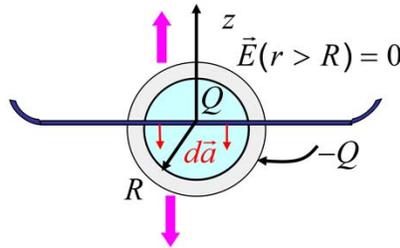
$$\Rightarrow \vec{F} = \hat{x} E_0 (\sigma \ell w) = \vec{E} Q$$

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Here is an example of the stress tensor where the force on the charge is a shear rather than a pressure. A shear force lies parallel to the area we are considering rather than perpendicular. We are going to calculate the force on a rectangular section of an infinite plane of charge due to a constant external field of E_0 in the x direction. The plane carries a charge density σ and is perpendicular to the z -axis and we are considering a rectangle of width w and length L . Since the charges within the rectangle cannot exert a force on themselves, it is fairly obvious that the net force on the charged rectangle is in the x -hat direction with a magnitude of $|F| = E_0 Q = E_0 (\sigma w L)$ but lets check that the stress tensor method gives this answer. To compute the force we need to integrate $T \cdot d\text{Area}$ over any surface that bounds the charge. In this case we use an infinitesimally thin box of width w and length L . Since all electrical fields are finite, the four sides with a height along the z -axis will contribute 0 force in the $\Delta z \Rightarrow 0$ limit and hence we only need to consider the top and bottom of the box. Since the force on the top and bottom surface is in the x -direction, it is indeed a strain rather than a pressure. The top and bottom area vectors are $+ dx dy \hat{z}$ for the top surface and $- dx dy \hat{z}$ for the bottom side using "outward" convention. The force components on either the top or bottom surface are just the T matrix times either area column vector. We compute the F_x component of the force which involves the $E_x E_z$ product where E_x is E_0 and E_z is due to the infinite plane of charge (which is $\pm \sigma / [2 \epsilon_0]$). Since both the area vector and E_z change sign going from the top to the bottom surface, both surfaces contribute the same force and we indeed get our expected F_x component which equals the external E -field times the charge within the rectangle. But can we show that $F_y = F_z = 0$? Lets consider F_z . This will be the x - y double integral of $E_z^2 - E_0^2$ (since there is no E_y). But $E_z^2 - E_0^2$ is the same for the top and bottom sides while their area vectors are equal and opposite hence F_z for the top will cancel F_z for the bottom. The $F_y=0$ argument is even simpler.

Split charge forces using a bounding plane

Force on north hemisphere of shielded charge using xy plane as bounding surface.



$$F_z = \int T_{zz} da_z$$

$$4\pi\epsilon_0 r^2 E_r = Q \left(\frac{r^3}{R^3} \right) \rightarrow \vec{E}(r < R) = \frac{Qr}{4\pi\epsilon_0 R^3} \hat{r}$$

on x-y plane $\vec{E}(s < R) = \frac{Qs}{4\pi\epsilon_0 R^3} \hat{s}$

$$T_{zz} = \frac{\epsilon_0}{2} (\mathcal{K}_z^2 - E_x^2 - E_y^2) = -\frac{\epsilon_0}{2} E_s^2 = -\frac{Q^2 s^2}{32\pi^2 \epsilon_0 R^6}$$

$$da_z = -2\pi s ds ; F_z = \int_0^R -\frac{Q^2 s^2}{32\pi^2 \epsilon_0 R^6} (-2\pi s ds)$$

$$F_z = \int_0^R \frac{Q^2 s^3}{16\pi\epsilon_0 R^6} ds = \frac{Q^2}{16\pi\epsilon_0 R^6} \left[\frac{s^4}{4} \right]_0^R$$

$$\rightarrow \vec{F}_N = \frac{Q^2}{64\pi\epsilon_0 R^2} \hat{z}$$

Presumably $\vec{F}_S = -\frac{Q^2}{64\pi\epsilon_0 R^2} \hat{z}$

15

In this problem we calculate the force on the northern hemisphere of a spherical charge distribution. We consider a uniform ball of charge Q with radius R which we shield by a thin shell of radius $-Q$. This restricts the E field to $r < R$. You consider the force on the isolated shell which only has an E -field when $r > R$ in homework. Often thinking of the xy plane as the bounding surface is a good choice when calculating the force on one half of a charge distribution on the other half. We can think of plane as a spherical surface which encloses the “northern” hemisphere in the limit of infinite radius. Presumably the force on the northern hemisphere lies along the z -hat direction and along the negative z -hat direction for the southern hemisphere. Our area vector is in outward direction from the northern hemisphere and is perpendicular to the xy plane which means it is in the $-z$ -hat direction. Since the force is along the z -hat direction and the area vector is along the $-z$ -hat direction we only need the T_{zz} component of the stress tensor. To calculate T_{zz} we need the E -field for the uniform ball of charge. As you recall, we can get this by Gauss’s law, where the enclosed charge which by a simple scaling argument is Q times the cube of the r/R ratio. This gives us an E -field which is proportional to r . We only need T_{zz} on the xy plane where $r \hat{r} = s \hat{s}$. This means $E_z=0$ and $E_x^2 + E_y^2$ is just E_s^2 . We set up the F_z integral in terms of T_{zz} and the area vector which is very easy to evaluate. The negative da times the negative T_{zz} gives a positive F_z . We know from Newton’s 3rd that the force on the southern hemisphere due to the northern hemisphere must lie in the opposite direction. In the stress tensor formalism, we get the negative sign since for the southern hemisphere, the outward area vector is along the positive z -hat direction, rather than the negative z -hat direction which applies to the northern hemisphere.

Field momentum

The force is the rate of change of momentum \vec{p}_{mech}

$$\vec{F} = \frac{d\vec{p}_{\text{mech}}}{dt} = \int_V \vec{f} \, d\tau = \int_S \vec{T} \cdot d\vec{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} \, d\tau$$

If we allow for a "field momentum" the last term suggests a natural

choice $\frac{d\vec{p}_{EM}}{dt} = \epsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S} \, d\tau \Rightarrow \vec{p}_{EM} = \epsilon_0 \mu_0 \int_V \vec{S} \, d\tau$ since

essentially all terms involve $\int d\tau$, simplest form involves

densities $\vec{p}_{EM} = \int_V \vec{P}_{EM} \, d\tau$ where $\vec{P}_{EM} = \epsilon_0 \mu_0 \vec{S} = \frac{\vec{S}}{c^2}$

Returning to $\int_S \vec{T} \cdot d\vec{a} = \int_V \nabla \cdot \vec{T} \, d\tau$ we have :

$$\frac{d\vec{P}_{EM} + \vec{P}_{\text{mech}}}{dt} = \nabla \cdot \vec{T} \quad \text{Comparing to} \quad \frac{d\rho}{dt} + \nabla \cdot \vec{J} = 0$$

we see $-\vec{T} \Leftrightarrow \vec{J}$ except $-\vec{T}$ describes momentum flow rather than charge flow.

Since $\vec{L} = \vec{r} \times \vec{p}$, $\vec{\ell} = \vec{r} \times \vec{P}_{EM} = \epsilon_0 \mu_0 \vec{r} \times \vec{S} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$

would be a good for field angular momentum density.

16

One good use of the stress tensor formalism is to compute the field momentum. The mechanical force (on the left side of the equation) can be thought of as the rate of change of the momentum of the volume of charge. We can think of the tensor surface integral as the total electromagnetic force. But what is the second term which is proportional to the rate of change of the Poynting's vector? A natural interpretation would be that this is the momentum carried by the electromagnetic field. This would be equivalent to saying the field tensor integral is the total force on the charge which is the rate of change of the mechanical momentum carried by the charge plus the electromagnetic field momentum. The idea of a field momentum may seem novel, but after all we know electromagnetic forces carry energy so why not momentum? We will see the inevitability of the implied relationship between the electromagnetic energy and momentum under the theory of relativity. Under this interpretation the surface integral of the stress tensor gives the rate of change of the "electromagnetic" momentum + the "mechanical" momentum. A somewhat more elegant approach is work in terms of momenta (volume) densities. The electromagnetic momentum density is proportional to the Poynting vector. The divergence of the stress tensor then gives the rate of change of the mechanical plus the electromagnetic momenta density. We note the similarity of this form to the continuity equation. In the continuity equation the divergence of the current density is related to the rate of change of the charge density. Here the divergence of the stress tensor is the rate of change of the momentum density. Hence $-\vec{T}$ is much like the current density \vec{J} except it is a more complicated object since it describes the flow of a momentum which is a vector rather than a charge which is a scalar. Since the angular momentum of a point particle with respect to an origin is the displacement vector crossed into its momentum, we can define a electromagnetic angular momentum density as the displacement vector crossed into the electromagnetic momentum density. Evidently an electromagnetic field contains both a linear momentum as well as an angular momentum as long as both the electric and magnetic fields are present.

Field angular momentum example

ℓ might explain how the rim of charge picks up $\vec{L} = \pi a^2 b \lambda B_0 \hat{z}$ when magnetic field is shut off due to Faraday EMF. Here is another example which is easier to calculate

We two long cylinders of Q and $-Q$ within solenoid
Calculate the torque on $+Q$ as \vec{B} changes.

$$\epsilon = -\frac{\partial \Phi}{\partial t} = -\frac{\partial(\pi a^2 B_z)}{\partial t} = -(\pi a^2) \frac{\partial B_z}{\partial t} = \oint \vec{E} \cdot d\vec{\ell} = 2\pi a E_\phi$$

$$E_\phi = -\frac{a^2}{2a} \frac{\partial B_z}{\partial t}; F_\phi = QE_\phi = -\frac{a^2 Q}{2a} \frac{\partial B_z}{\partial t}$$

$$\vec{N}_a = \vec{r} \times \vec{F} = -a \times \frac{Qa^2}{2a} \frac{\partial B_z}{\partial t} \hat{z} = -\frac{Qa^2}{2} \frac{\partial B_z}{\partial t} \hat{z}$$

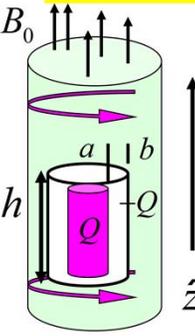
Similarly $\vec{N}_b = +\frac{Qb^2}{2} \frac{\partial B_z}{\partial t} \hat{z} \rightarrow \vec{N} = \frac{Q(b^2 - a^2)}{2} \frac{\partial B_z}{\partial t} \hat{z}$

Start with $\vec{B}(0) = B_0 \hat{z}$ and slowly turn off B : $\vec{B}(\infty) = 0$

$$\vec{L}_Q = \int_0^\infty \vec{N} dt = -\frac{Q(b^2 - a^2)}{2} \hat{z} \int_{B_0}^0 dB' = -\frac{Q(b^2 - a^2) B_0}{2} \hat{z} \quad 17$$

As abstract and implausible the concept of an angular momentum density is, we really need it to preserve angular momentum conservation in the presence of Faraday's law. Recall last chapter's example of rim of charge in the presence of a collapsing magnetic field. As we argued before a changing magnetic field creates an EMF which in this case will create a force and torque on charged rim. In the absence of any external forces such as friction, this torque will change the angular momentum of the rim. What other system changes angular momentum to compensate for the rim angular momentum? One hint is that, the angular momentum is independent of the details on how the field is collapsed but only depend on the charge density, the geometry, and the initial strength of the B-field. This strongly suggests that the rim acquires the angular momentum that was initially stored in the crossed electric field (due to the charge) and the magnetic field (provided by the magnet). In order to verify the concept of the field angular momentum being transferred to a charged object as the field is collapsed, we will use a simpler geometry where the electric and magnetic fields can easily be computed. This problem is a slightly simplified version of a Griffiths example. We will place oppositely charged coaxial cylinders of radius "a" and b and length h in the field due to an infinite solenoid of radius R which initially contains a field of B_0 . We use two oppositely charged cylinders to completely contain the electric field to the region $a < s < b$ so that we can reliably compute the field angular momentum and avoid fringe fields and to kill any $\text{Del} \cdot \vec{T}$ terms outside of the two charged cylinders which would indicate that the two cylinders system is not isolated but field momentum can flow in. We start by computing the torque on the inner cylinder with $+Q$. We compute the EMF using the rate of change of flux within the surface of the inner cylinder where the charge resides and set this equal to the circumference times the E-field. The product of the E-field, the charge, and the transverse distance from the charge to the center of rotation "a" gives us torque expression. This is a somewhat informal way of computing the torque since we are computing the torque relative to the z-axis rather than with respect to a formal reference point such as the origin. We illustrate a more formal approach on the next slide. Our torque expression assumes we are in the quasi-static limit since a rapidly changing B can induce E which add a displacement contribution to our solenoidal magnetic field. We can "recycle" the inner cylinder torque expression to get the torque on the outer cylinder by changing "a" to b and Q to $-Q$. We then integrate the total torque expression to get the angular momentum of the two cylinder system when the initial B_0 is **slowly** turned to zero (to keep us in the quasi-static limit). The cylinder system rotates against the indicated arrow as the field is turned off. This means the cylinders acquire an angular momentum along the $-\hat{z}$ direction.

Did L_Q reside in fields?



Does $\vec{L}_Q = \frac{Q(b^2 - a^2)B_0}{2} \hat{z} = \int_V \vec{\ell} d\tau = \epsilon_0 \int_V \vec{r} \times (\vec{E} \times \vec{B}_0) d\tau$??

We only have both E and B from $a < s < b$ & $|z| < h$

$\vec{E} = \frac{Q/h}{2\pi\epsilon_0 s} \hat{s}$; $\vec{E} \times \vec{B}_0 = \frac{QB_0}{2\pi\epsilon_0 sh} \hat{s} \times \hat{z} = -\frac{QB_0 \hat{\phi}}{2\pi\epsilon_0 sh}$; $\vec{r} = z \hat{z} + s \hat{s}$

$\vec{\ell} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}_0) = -\frac{QB_0}{2\pi sh} (z \hat{z} + s \hat{s}) \times \hat{\phi} = -\frac{QB_0}{2\pi sh} (s \hat{z} - z \hat{s})$

Thus $\vec{\ell} = -\frac{QB_0}{2\pi sh} (s \hat{z} - z \hat{s}) = -\frac{QB_0}{2\pi sh} (-z \cos \phi \hat{x} - z \sin \phi \hat{y} + s \hat{z})$

$$d\tau = s ds d\phi dz \rightarrow \vec{L} = -\frac{QB_0}{2\pi h} \int_{-h/2}^{h/2} dz \int_a^b \frac{s ds}{s} \int_0^{2\pi} d\phi (-z \cos \phi \hat{x} - z \sin \phi \hat{y} + s \hat{z})$$

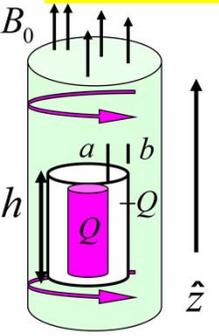
$$\vec{L} = -\frac{QB_0}{2\pi h} \hat{z} \int_{-h/2}^{h/2} dz \int_a^b \frac{s ds}{s} 2\pi s = -\frac{QB_0 \hat{z}}{h} \int_a^b ds \left[\frac{b^2 - a^2}{2} \right] = -\frac{QB_0}{2} (b^2 - a^2) \hat{z}$$

Hence angular momentum is conserved, field angular momentum is real, and all is right with the world! 18

We next compute the total angular momentum that resided in the fields prior to turning off the magnet by integrating the angular momentum density. The angular momentum density involves the displacement vector to a reference point times momentum density which is essentially E cross B. In general the angular momentum depends on the reference point which we take as the origin. The z component of the angular momentum density is a constant, but there is an s-hat term which isn't. This s-hat term will vanish since its Cartesian coordinates are proportional to cosine and sine of phi which will vanish after integrating over phi which is part of the volume integral. We formally work out the volume integral of the angular momentum density as a triple integral over z, s, and phi. It is important to use Cartesian unit vectors which are constants when doing such integrals. We indeed find that field angular momentum equals the angular momentum created by the torque as the B-field collapses.

Our result is just the z-component of the angular momentum density times the volume of the thick cylindrical shell. The constant z-component density is easy to understand. The density is just r cross E cross B. E falls as 1/s for a charged cylinder and the 1/s is canceled by the z-component of the vec r factor which is s times s-hat. Hence the z-component of the total angular momentum is just the density times the volume between the two cylinders of height h. We get exactly the same initial field angular momentum that goes into the two cylinder system created by the EMF when the B_0 field is shut off. Hence it looks like field angular momentum restores angular momentum conservation to Faraday's law. Griffith's shows that this works even for the case where the +Q shell is within the solenoid and the -Q shell is outside of the solenoid.

Is **linear** momentum stored in the fields?



$$\hat{s} \times \hat{z} = -\hat{\phi}$$

We only have both E and B from $a < s < b$ & $|z| < h$

$$\vec{P}_{EM} = \epsilon_0 (\vec{E} \times \vec{B}); \quad \vec{E} \times \vec{B} = \frac{QB_0}{2\pi\epsilon_0 sh} \hat{s} \times \hat{z} = -\frac{QB_0\hat{\phi}}{2\pi\epsilon_0 sh}$$

Don't try to move $\hat{\phi}$ through the integral and get the wrong answer:

~~$$\vec{P}_{EM} = -\frac{QB_0}{2\pi h} \int_{-h/2}^{h/2} dz \int_a^b \frac{2\pi s ds}{s} \hat{\phi} = -\frac{QB_0\hat{\phi}}{2\pi h} \int_{-h/2}^{h/2} dz \int_a^b \frac{2\pi s ds}{s} = -\frac{QB_0\hat{\phi}}{2} (b-a)$$~~

Since $\hat{\phi}$ depends on ϕ use triple integral form: $d\tau = s ds d\phi dz$

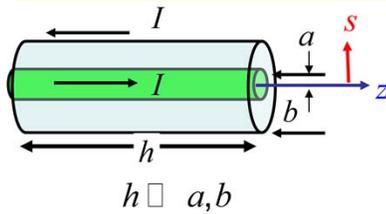
$$\vec{P}_{EM} = \int \vec{P}_{EM} d\tau = -\frac{QB_0}{2\pi h} \int_{-h/2}^{h/2} dz \int_a^b \frac{s ds}{s} \int_0^{2\pi} d\phi (-\sin\phi \cos\phi \ 0)_\phi = \vec{0}$$

Makes sense since $d\vec{F} \propto \hat{\phi} \rightarrow \vec{F} \propto \int d\vec{F} = 0$

19

We decided that as the B field is lowered, a torque is set up which causes the two cylinders to rotate and hence there must be an angular momentum stored in the fields which we calculated on the last slide. But is there a linear momentum? We start off with the linear momentum density which is proportional to E cross B which points in the phi direction. To get the total angular momentum we integrate over the relevant volume which is the thick shell between $s=a$ and $s=b$ and length h since this is the only region with a non-zero momentum density. We start with the **wrong way** of finding P where we use partially integrated form for the volume integral like $d\tau = 2\pi s ds dz$ and attempt to move the phi-hat unit vector past the integral to get a P along phi-hat. The problem with this is that phi-hat is not a constant unit vector but depends on phi meaning we must use a fully differential volume element in $ds d\phi dz$ and express phi-hat in terms of constant, Cartesian unit vectors. The phi integral causes P to vanish and hence we conclude that P vanishes. This makes sense since the collapsing B-field creates an azimuthal force which integrates to zero. Hence this geometry gives us an angular momentum but no linear momentum.

Revisiting the coaxial cable



Recall our Poynting Vector example

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{I}{2\pi s} \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \times \hat{\phi} = \frac{\lambda I}{\epsilon_0 (2\pi)^2 s^2} \hat{z}$$

$$\text{where } \int_s \vec{S} \cdot d\vec{a} = \text{power} = \frac{\lambda I}{2\pi\epsilon_0} \ln \frac{b}{a} = VI$$

$$\vec{\mathcal{P}}_{EM} = \epsilon_0 \mu_0 \vec{S} = \frac{\mu_0 \lambda I}{(2\pi)^2 s^2} \hat{z} \Rightarrow \vec{p}_{EM} = \mu_0 h \int_a^b 2\pi s \vec{\mathcal{P}}_{EM} ds$$

$$\vec{p}_{EM} = \frac{\lambda I h \mu_0}{2\pi} \int_a^b \frac{s}{s^2} ds = \hat{z} \frac{\mu_0 \lambda I h}{2\pi} \ln \frac{b}{a}$$

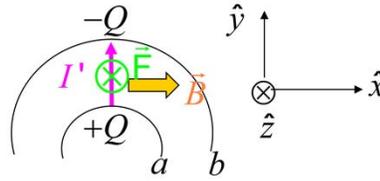
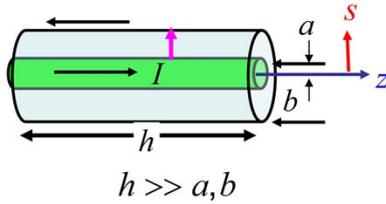
But this is entirely static! The cable is clearly not moving. Where does the momentum canceling \vec{p}_{EM} reside?? It is a “hidden” momentum carried by the moving charges that constitute the currents and is a relativistic effect. I would say it is a **well-hidden momentum!**

20

Our treatment of the stress tensor also suggests that there will be a linear momentum density contained in the electromagnetic fields in regions where both E and B are present. This linear momentum density will be proportional to the Poynting vector. The coaxial cable provided an example of a system with a Poynting vector that points along the direction of the inside current. Including the μ_0 ϵ_0 factor we get an angular momentum density that also points in the inside current direction. If we integrate the momentum density over a field volume for a length h of cable we get a form proportional to the current, the charge density, and the logarithm of radius ratio which is very similar to the voltage times current power expression. But the momentum carried by the cable is very small for practical cables since it is down by a factor of $\epsilon_0 \mu_0$ compared to the power -- which is a factor of c^2 or about 10^{-17} . It is reassuring that our typical cables carry a fairly small amount of momentum. It's good that we don't have to chase our cable TV around the house. But why is there any momentum at all? Everything is static (eg constant charges and constant currents). In the case of the two cylinders, the field angular momentum compensates for the angular momentum created by the torque created by Faraday's law – does the field momentum compensate for some other “missing” momentum to keep the cable from moving? About the only source of hidden momentum that one can think of would be the momentum of the moving charges that create the current. Ultimately this is the source of the hidden momentum which is compensated by the field momentum but the argument is very subtle and is basically a relativistic effect which we might discuss later. As one expects, relativistic effects are very small at every day velocities.

Extracting the “hidden” momentum

Discharge coax by inserting radial **wire**



The **wire** will carry I' and a magnetic force will push wire along z axis.

$$d\vec{F}_{\text{wire}} = I' d\vec{\ell} \times \vec{B}; \quad d\vec{\ell} = ds \hat{s}; \quad \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \Rightarrow d\vec{F} = \hat{s} \times \hat{\phi} \frac{\mu_0 I I'}{2\pi s} ds$$

$$d\vec{F} = \hat{z} \frac{\mu_0 I I'}{2\pi s} ds \Rightarrow \vec{F}_{\text{wire}} = \int_a^b d\vec{F} = \hat{z} \frac{\mu_0 I I'}{2\pi} \int_a^b \frac{ds}{s} = \hat{z} \frac{\mu_0 I I'}{2\pi} \ln \frac{b}{a}$$

$$\vec{p}_{\text{wire}} = \int_0^\infty F dt = \left(\hat{z} \frac{\mu_0 I}{2\pi} \ln \frac{b}{a} \right) \int_0^\infty I' dt = \hat{z} \frac{\mu_0 I \Delta Q}{2\pi} \ln \frac{b}{a}$$

$$\Delta Q = \lambda h \rightarrow \vec{p}_{\text{wire}} = \hat{z} \frac{\mu_0 I \lambda h}{2\pi} \ln \frac{b}{a} = \vec{p}_{\text{EM}}$$

21

However we can extract the “hidden momentum” and “measure” its momentum. Griffiths Ex 8.3 describes the extraction of the momentum by collapsing the magnetic field by turning off the current. Here is even a simpler way. Imagine we drain the charge by shorting the coax with a radial wire. A current, which we will call i' , will flow from the inner to the outer conductor which is radially outward. This current is transverse to the magnetic field created by the current through the inner conductor i and thus the wire feels a magnetic force along the z -hat direction. We can find the total force by integrating $i' \mathbf{B} \cdot d\mathbf{l}$ and find that the z -force on the wire is proportional to $i' \ln(b/a)$. The total impulse delivered to the wire is then the integral of $F dt$ which is proportional to the integral of $i' dt$ or the total charge Q that flows from the inner to the outer conductor. We find the impulse delivered to the wire when all the charge is extinguished exactly equals the total momentum that was initially in the crossed \mathbf{E} and \mathbf{B} fields. After the charge is extinguished the \mathbf{E} field vanishes and there is no field momentum – it all got transferred to the wire. If we could figure out a way to make a friction free, conducting wire connection to both cylinders, this momentum would flow into the wire and give it a velocity such that $m\mathbf{v} = \mathbf{p}_{\text{field}}$. The Faraday’s law approach used by the text is generally a safer approach than our method of shorting the charge since the magnetic field might interact with the currents created as the conductor charge travels towards the shorting wire. In this case the magnetic force that affects P_z would be in the z -hat direction, the magnetic field is in the ϕ -hat direction, so to contribute to P_z any surface current must be in the s -hat direction which is impossible for the cylinders that comprise the coax since these surfaces are at fixed $s=a$ or $s=b$. The only s -hat currents flow on the shorting wire.