Homework #3

1. Consider a plane polarized electromagnetic wave with an electrical field given by $\vec{E} = \hat{x}\ E_0 \cos(\omega t - kz)$.

(a) Find the associated magnetic field

(b) Find the energy density $u_{em}\left(z,t\right) = \frac{\varepsilon_0 \vec{E} \bullet \vec{E}}{2} + \frac{\vec{B} \bullet \vec{B}}{2\mu_0}$

(c) Find the Poynting vector $\vec{S}(z,t)$. Do not average over time.

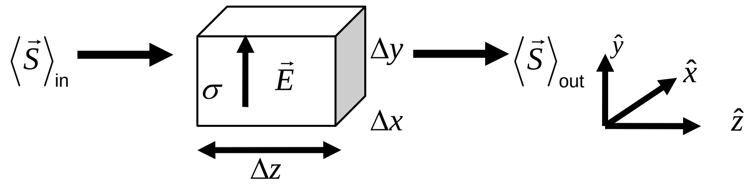
(d) Use your answers for (b) and (c) to show $\vec{\nabla} \bullet \vec{S} + \frac{\partial u_{em}}{\partial t} = 0$

(e) Use your answer for (c) and relations such as $\left\langle \cos^2 \omega t \right\rangle = \frac{1}{2}$ etc to show $\left\langle \vec{S} \right\rangle = \frac{E_0^2}{2\mu_0 c}\hat{z} \approx \frac{E_0^2 \hat{z}}{2 \times 377 \text{ Ohms}}$. To get full credit be sure to demonstrate that $\mu_0 c$ has the units of impedance or Ohms and be careful to show $\left\langle \cos^2 \omega t - kz \right\rangle = \frac{1}{2}$ independent of z

(f) Show you get the same result for $\left\langle \vec{S} \right\rangle$ as in part (e) using complex notation: $\vec{E} = \hat{x} E_0 \exp\left(ikz - i\omega t\right)$ and $\left\langle \vec{S} \right\rangle = \frac{1}{2\mu_0} \text{Re}\left\{\vec{E}^* \times \vec{B}\right\}$. Assume E_0 is real. Begin by computing the complex $\vec{B}$ using Faraday' law assuming $\vec{B} \propto \exp\left(-i\omega t\right)$. What is $\vec{\nabla} \bullet \left\langle \vec{S} \right\rangle$? Is your $\vec{\nabla} \bullet \left\langle \vec{S} \right\rangle$ equal to $\vec{\nabla} \bullet \vec{S}$ of part (d)?

2. Griffiths problem 9.12 . Use the time averaged $\left\langle \vec{T} \right\rangle$ throughout.

3. Consider an electromagnetic wave with a time-averaged $\left\langle \vec{S} \right\rangle$ along the $\hat{z}$ direction. Assume this wave passes through an infinitesimal, conducting volume with conductivity σ, and dimensions $\Delta x, \Delta y, \Delta z$. Assume that the wave's electrical field while in the volume is given by $\vec{E} \propto \hat{y}$ as shown below:



(a) Write the total time-averaged power flowing into the volume in terms of $\langle\vec{S}_{\text{in}}\rangle_z$, $\langle\vec{S}_{\text{out}}\rangle_z$,and $\Delta x\ \Delta y$ as shown above.

(b) The time-averaged. power dissipated in heat within the volume is presumably: $\langle I^2\rangle R$. Write the average power dissipated in heat in terms of the field within the volume $\vec{E}$, the conductivity σ , and the volume dimensions $\Delta x, \Delta y, \Delta z$. Hints – You can find I from $\vec{J} = \sigma\vec{E}$ and the resistance R can be computed from σ and the dimensions $\Delta x, \Delta y, \Delta z$

(c) By setting the time-averaged power flowing into volume that you computed in part (a) to the heat dissipated power that you computed in part (b) show that : $\vec{\nabla}\bullet\langle\vec{S}\rangle + \langle\vec{E}\bullet\vec{J}\rangle = 0$ which is the time-averaged version of Poynting's theorem.

Hint-- $\left[\langle\vec{S}_{\text{in}}\rangle_z - \langle\vec{S}_{\text{out}}\rangle_z\right] \propto \partial\langle S\rangle_z / \partial z$ in the infinitesimal limit.

4. A superposition of two electromagnetic waves traveling in vacuum has an electric field of the form $\vec{E} = E_0\hat{z}\,e^{ikx-i\omega t} + E_0\hat{z}\,e^{iky-i\omega t}$. For simplicity assume E_0 is real.

(a) Find the associated magnetic field using Faraday's law.

(b) Construct the time-averaged Poynting vector using $\langle\vec{S}\rangle = \dfrac{\text{Re}\left\{\vec{E}^* \times \vec{B}\right\}}{2\mu_0}$

(c) Compute the time-averaged intensity $I = \left|\langle\vec{S}\rangle\right|$. Write the maximum possible intensity in terms of E_0. Crudely sketch $I(x, y=0, z=0)$ and $I(x=y, z=0)$ as a function of x. These are the intensities on the x axis and on the x=y line.

(d) Find $\vec{\nabla}\bullet\langle\vec{S}\rangle$ using your answer to part (b). Did you get what you expect from Poynting's Theorem?