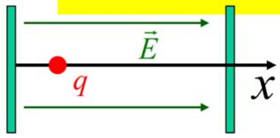
Relativistic Dynamics and Electrodynamics

- Newton's law and Lorentz law are still ok with relativity
  - F= dp/dt holds relativistically
  - You just need to use $p = m\,v\,\gamma$
  - F=q(E + v × B) is ok relativistically as well
- Work energy theorem also ok with relativity
- Many important E&M quantities are 4-vectors
  - ρ and J
  - V and A
- The E and B fields are not parts of 4-vectors but are components of an asymmetric tensor.
- Many laws can be written as dot-product invariants:
  - Continuity Eqn.
  - Lorentz Gauge
- Gauss's law and Ampere's law are ok relativistically as are rest of Maxwell's equations
- The velocity dependence of magnetic forces is basically a relativistic effect

1

We end the course with this chapter describing electrodynamics in relativistic language. It will also be our first chapter on relativistic dynamics as oppose to the relativistic kinematic kinematics which has been our focus so far. Many of the results you are familiar with in physics still "work" in relativity with little or no modification. For example, Newton's second law still hold in the form F= dP/dt but P is not mV but rather gamma m v. The work-energy works as well. This chapter dramatically enlarges the number of 4-vector objects over our kinematic chapters. Now the 4-interval and 4-momentum is augmented with 4-potentials, 4-currents, and 4-gradients and many important relations such as the continuity equation and the Lorentz gauge can be written as elegant 4-vector dot product expressions using these 4-vectors. Interestingly enough, the electromagnetic fields cannot be combined as 4-vectors but rather form parts of a 4 by 4 asymmetric "field"-tensor. Our main conclusion is that electrodynamics is already "relativistically ready". In crude language: whereas almost everything you learned in elementary mechanics was wrong, nearly everything you learned in elementary electrodynamics was right.

Hyperbolic motion



$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{E} + q\vec{u} \times \vec{B}$ = is ok relativistically if one uses $\vec{p} = \gamma m \vec{u}$

Consider 1D motion in a uniform $\vec{E}$

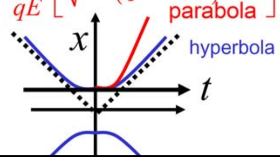
$$F\Delta t = \Delta p;\ qEt = p = \gamma m u = \frac{mu}{\sqrt{1-(u/c)^2}} \rightarrow \left(\frac{qEt}{m}\right)^2 \left(1 - \left(\frac{u}{c}\right)^2\right) = u^2 \rightarrow u^2\left[1 + \left(\frac{qEt}{mc}\right)^2\right] = \left(\frac{qEt}{m}\right)^2$$

$$u = \frac{qEt/m}{\sqrt{1+\left(\frac{qEt}{mc}\right)^2}} \xrightarrow{t\rightarrow\infty} c \quad \text{Find x(t);} \quad \frac{dx}{dt} = \frac{c\left(qEt/mc\right)}{\sqrt{1+\left(\frac{qEt}{mc}\right)^2}} \rightarrow x = c\int_0^t \frac{qEt'/mc}{\sqrt{1+\left(qEt'/mc\right)^2}}dt'$$

$$\nu = 1 + \left(qEt'/mc\right)^2 \ ;\ d\nu = 2\frac{qE}{mc}\left(qEt'/mc\right)dt' \ ;\ \nu_{\text{up}} = 1 + \left(qEt/mc\right)^2 \ ;\ \nu_{\text{low}} = 1$$

$$x = \frac{mc^2}{2qE}\int_{\nu_{\text{low}}}^{\nu_{\text{up}}} \nu^{-1/2}d\nu = \frac{mc^2}{qE}\left[\sqrt{\nu}\right]_1^{1+\left(qEt/mc\right)^2} = \frac{mc^2}{qE}\left[\sqrt{1+\left(qEt/mc\right)^2} - 1\right] \text{ or } \left(\frac{qEx}{mc^2}+1\right)^2 - \left(\frac{qEt}{mc}\right)^2 = 1$$

$$x = \frac{mc^2}{qE}\left[\sqrt{1+\left(qEt/mc\right)^2} - 1\right] \xrightarrow{qEt/mc \ll 1} \frac{mc^2}{qE}\left[1 + \frac{1}{2}\left(\frac{qEt}{mc}\right)^2 - 1\right] \rightarrow \frac{qE}{2m}t^2$$

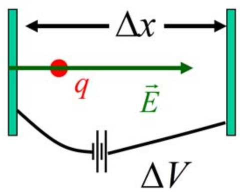


$x = \frac{qE}{2m}t^2$ or parabolic motion is a NR approx to the relativistically correct hyperbolic motion

2

Thus far we have been considering relativistic kinematics. In this chapter we begin by describing how a point charge responds to an electromagnetic field. The Lorentz force which describes the electric and magnetic force is still correct relativistically as we will show in depth later in this chapter. We will write the particle velocity as u-vec to correspond to Griffiths and to keep it different from v-vec which Griffiths uses for frames. Newton was basically right, the force is the rate of change of momentum. The force is the force in the given (e.g. lab frame) and time is the time in lab frame – not the particle frame. The only difference with the non-relativistic dynamics-- that you know and love -- is that one must use the relativistic form for the momentum or you can get a charged particle to exceed the speed of light, but this cannot happen if one uses the relativistic form because of the gamma factor as we will show. As our first dynamic problem, we consider the case where there is a constant electric field along the x-axis (say from a large capacitor) and no magnetic field. We will consider the case of a constant magnetic field shortly. Newton's 2nd law says the change in the momentum is the force times time (i.e. impulse). For simplicity we will start the particle from rest in the lab frame. This means the momentum is q E t. We need to do a little more algebra to compute the velocity u at this time. We note that as t approaches infinity u approaches (but does not exceed) the speed of light. To get x(t) we need to integrate our velocity expression with respect to time. Fortunately the integral is easy to do and we get a reasonably simple form for x(t). We rearrange this answer to make it clear that a plot of x versus t is a hyperbola (hence the term hyperbolic motion). As shown in the sketch, a hyperbola asymptotically approaches a straight line which in this case is x = c t. One important check is to go to the non-relativistic limit. We expand the result to lowest non-vanishing order in qEt/mc and find x = ½ (qE/m)t^2 which non-relativistically is ½ a t^2 where a = F/m. As shown in the sketch, the non-relativistic parabola looks matches the hyperbola at low times, but the slope for the parabola (i.e. speed) incorrectly approaches infinity at long times.

The work-energy theorem works as well



The charge moves from $x = 0$ to $x = \frac{mc^2}{qE}\left[\sqrt{1+\left(qEt/mc\right)^2} - 1\right]$

Hence $\Delta x = x$ and $qE\Delta x = mc^2\left[\sqrt{1+\left(qEt/mc\right)^2} - 1\right]$

We can write $\left[\sqrt{1+\left(qEt/mc\right)^2} - 1\right]$ in terms of $u = \frac{qEt/m}{\sqrt{1+\left(\frac{qEt}{mc}\right)^2}}$

$$1-\frac{u^2}{c^2} = \frac{1}{1+\left(\frac{qEt}{mc}\right)^2} \rightarrow \sqrt{1+\left(qEt/mc\right)^2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma$$ Thus $qE\Delta x = mc^2\left(\gamma - 1\right) = \Delta T = q\Delta V$

We can also apply this for infinitesimal Δx to get power expression

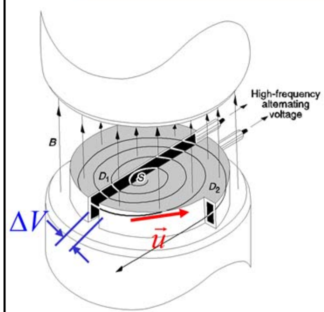
$\vec{F}\bullet\Delta\vec{r} = \Delta T \rightarrow \vec{F}\bullet\frac{d\vec{r}}{dt} = \frac{dT}{dt} \rightarrow$ power $= P = \vec{F}\bullet\vec{u}$. Lets check this.

$$P = \vec{u}\bullet\vec{F} = \vec{u}\bullet\frac{d\vec{p}}{dt} = \vec{u}\bullet\frac{d}{dt}\left[\frac{m\vec{u}}{\sqrt{1-(u/c)^2}}\right]_{\vec{p}} = \frac{d}{dt}\left[\frac{mc^2}{\sqrt{1-(u/c)^2}}\right] = \frac{d\mathcal{E}}{dt} = \frac{dT}{dt}$$

3

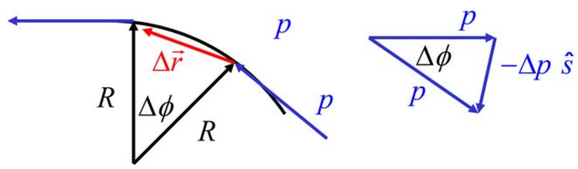
Not only does F= delta P/delta t hold relativistically but delta KE = vec-F dot delta vec-r works as well. We can show this from our previous work on hyperbolic motion. The force acting on the particle is a qE and it acts over a distance of x. Inserting our expression for x(t), and re-writing our x expression in terms of u we find qE x is just mc^2(gamma -1) which is the total energy minus the rest energy which is the kinetic energy. We can also write the KE as the change of the voltage times the charge – just like you learned in Physics 212. We can also show this from the power expression power = vec-F dot vec-u. Again this looks like a NR result but hold relativistically as well We replace vec-F by the time derivative of p-vec. We can do considerable calculus to write this as the time derivative of gamma mc^2 which is the rate of change of the total energy which equals the rate of change of the kinetic energy since the total energy is just the KE plus mc^2.

Death of the cyclotron



$$\Delta|\vec p| = p\Delta\phi \Rightarrow \frac{d\,\vec p}{dt} = -p\frac{d\phi}{dt}\hat s$$

$$\left|\frac{\Delta\vec r}{\Delta t}\right| = u = R\frac{\Delta\phi}{\Delta t} \to \frac{d\phi}{dt} = \frac{u}{R} \to \frac{d\,\vec p}{dt} = -p\frac{u}{R}\hat s$$



$$\frac{d\,\vec p}{dt} = \vec F = q\vec u \times \vec B = qu\hat\phi \times B\hat z = -quB\hat s = -p\frac{u}{R}\ \hat s$$

$\Rightarrow \boxed{p = qBR}$ like NR case except $p = mu\gamma \to u = \frac{p}{m\gamma}$

$$\omega = \frac{2\pi}{\tau}\ ;\ \tau = \frac{2\pi R}{u} \Rightarrow \omega = \frac{u}{R} = \left(\frac{qBR}{m\gamma}\right)\frac{1}{R} = \frac{qB}{m\gamma} = \frac{\omega_{\text{cyc}}}{\gamma} = \frac{\omega_{\text{cyc}}}{E/mc^2} = \frac{\omega_{\text{cyc}}}{(1+T/mc^2)}$$

Hence as T $\to mc^2$; $\omega < \omega_{\text{cyc}}$ and D oscillation is no longer in phase. Thus cyclotron no longer accelerates. Bad news for E.O. Lawrence.

4

Previously we considered the motion of a charge in a constant electric field. Now we consider the motion of a charged particle in a constant magnetic field. The idea here is the charge is traveling in a circle with a tangential velocity of u. We locate the charge with a radius R and coordinate phi. The sketch shows that if the momentum is tangential to the circle, it changes by an amount p delta phi and the change is directed in the –s-hat (or centripetal) direction. Since u = R dphi/dt, we can write the rate of change of momentum as p u/R which is again directed centripetally (along –s-hat). The change in (relativistic) momentum is given by the Lorentz force or q vec-u cross B = -q u B s-hat. Setting this force equal to the rate of change of momentum gives us p = qBR after cancelling a common factor of u. This is **exactly** the same formula as we had non-relativistically. The only difference is we need to use the relativistic momentum or p = gamma m vec u. We can write the angular frequency of the charge as 2 pi/tau where tau is the period of rotation (**not the proper time**) and the period is the circumference over the speed u. Solving for R using p = qBR=m u gamma we get the usual cyclotron frequency divided by gamma which can be written in terms of the kinetic energy T. Hence for T<<mc^2, the orbital frequency is just the cyclotron frequency and a constant (Dee) frequency cyclotron can keep in phase with the particle and continue to accelerate it. But as T starts to approach mc^2, the oscillation becomes smaller than cyclotron frequency, the push of the D-field is out of synch and the situation is like a randomly pushing a child on a swing – no energy is transferred. The cyclotron no longer works!

Constructing 4-vectors

Use $\tilde{r} = \left(ct \quad \vec{r}\right)$ as a template

1) related scalar + vector 2) name after vector 3) powers of c for dimension

Momenta: $\mathcal{E},\vec{p}$ $\left[\frac{p}{\mathcal{E}} = \frac{mv}{mv^2/2} = \frac{1}{v}\right]_{units} \Rightarrow \tilde{p} = \left(\frac{\mathcal{E}}{c} \quad \vec{p}\right)$

Sources: $\rho,\vec{J}$ $\left[\frac{J}{\rho} = v\right]_{units} \to \tilde{J} \Rightarrow \left(c\rho \quad \vec{J}\right)$

Potentials: $V,\vec{A}$ $\left[\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \to \frac{\Delta A/\Delta t}{\Delta V/\Delta x} = 1 \to \frac{A}{V} = \frac{1}{v}\right]_{units} \Rightarrow \tilde{A} = \left(\frac{V}{c} \quad \vec{A}\right)$

Derivatives: $\frac{\partial}{\partial t},\vec{\nabla}$ $\left[\frac{\partial/\partial x}{\partial/\partial t} = \frac{1}{v}\right]_{units} \Rightarrow \tilde{\nabla} = \left(\frac{1}{c}\frac{\partial}{\partial t} \quad -\vec{\nabla}\right)$

But why $-\vec{\nabla}$? $\Delta f = \tilde{\nabla} f \bullet d\tilde{r} = \left(\frac{1}{c}\frac{\partial f}{\partial t} \quad -\vec{\nabla} f\right) \bullet \left(cdt \quad d\vec{r}\right) = dt\frac{\partial f}{\partial t} + d\vec{r} \bullet \vec{\nabla} f$

Stay tuned for how to write $\vec{E}$ and $\vec{B}$

5

We can think of the space-time interval as a template for building the necessary 4-vectors for electrodynamics. We construct the 4-vectors from related scalars and 3-vectors. For the space-time interval the 4 components are the "coordinates" that locate an event. We usually name the 4-vector after the vector component by replacing arrow with a tilde so r-vec becomes r-tilde. Finally we throw in powers of c in the 0 component so that the scalar and vector pieces have the same units (for space-time this is meters). A familiar example is the 4-momenta. The scalar is energy and the vector is momenta. In non-relativistic mechanics, the KE has mass times velocity square units while the momentum has mass times velocity units. Hence for dimensional consistency we need to divide the 0 component (energy) by a velocity (c) to get the 4-vector momentum. Another natural grouping is rho and J which are the sources terms in Maxwell's Equation for the electric and magnetic fields. Since J = rho v we use one power of c to the scalar rho for dimensional consistency with J. Another natural grouping is to combine the scalar potential with vector potential to create the 4-vector potential. Here unit ratio between A and V is less obvious. We get our clue by writing the E-field in terms of the scalar and vector potential. This tells us that the ratio of the gradient of V and the time derivative of A must be dimensionless. Hence A/V must have dimensions of 1/velocity and we restore dimensional consistency by combining V/c with A into a 4-potential. Finally we will want to combine the time derivative with the gradient to form a derivative 4-vector. Here the dimensions are fairly obvious, but include a unexpected minus sign. Evidently the "natural" derivatives (with no minus signs) is the **covariant** derivative and our usual contravariant 4-vector has a minus sign – why? The use of a covariant derivative allow us to write an infinitesimal change of a scalar function f as an invariant 4-doc product of its 4-gradiant and the differential of the space-time 4-vector. If we think of f as **scalar** function of coordinates -- it makes sense that a change in f is invariant under coordinate change.

The fact that we can combine related vectors and scalars into a 4-vector by simply adding factors of c (the invariant that led to relativity) strongly hints (eg **screams**) that the world is fundamentally relativistic . But this begs the question – how can E and B (at the heart of electrodynamics) combined into a relativistic object. Clearly there is no room in a 4-vector since it only has four slots. The trick is to combine E and B into a asymmetric four by four matrix called the field tensor which we will introduce shortly.

Fun with four-vectors

In analogy with

$\vec{p} = m\vec{v} \Rightarrow \tilde{p} = m\tilde{\eta} = m\begin{pmatrix} c\gamma & \vec{u}\gamma \end{pmatrix}$ we write

$\vec{J} = \rho\vec{u} \Rightarrow \tilde{J} = \rho_0\tilde{\eta} = \rho_0\begin{pmatrix} c\gamma & \gamma\vec{u} \end{pmatrix}$. Compute $|\tilde{J}|$

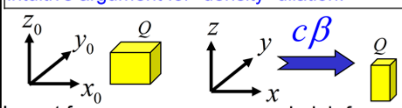
$\tilde{J} \bullet \tilde{J} = (\rho_0)^2 \gamma^2 (c^2 - u^2) = (c\rho_0)^2 \gamma^2 (1^2 - \beta^2) = (c\rho_0)^2$

Evidently ρ_0 is an invariant related to usual ρ.

In the NR limit $u << c$, $\gamma \to 1$; $\tilde{J} \to \begin{pmatrix} c\rho_0 & \rho_0\vec{u} \end{pmatrix}$

Since $\vec{J} = \rho\vec{v}$ we can think of ρ_0 as a "proper" density or charge density in the charge rest frame.

If we write $\tilde{J} = \begin{pmatrix} c\rho & \rho\vec{u} \end{pmatrix}$ even at high velocities we have $\begin{pmatrix} c\rho & \rho\vec{u} \end{pmatrix} = \rho_0\begin{pmatrix} c\gamma & \gamma\vec{u} \end{pmatrix} \to \rho = \gamma\rho_0$, similar to of dilation of proper time $\Delta t = \gamma\,\Delta\tau$. Below is an intuitive argument for "density" dilation:



In rest frame $\qquad\qquad$ In lab frame

$$\rho_0 = \frac{Q}{\Delta x_0 \Delta y_0 \Delta z_0} \qquad \rho = \frac{Q}{\Delta x \Delta y \Delta z}$$

$\rho = \dfrac{Q}{\Delta x \Delta y \Delta z}$ but $\begin{matrix} \Delta y = \Delta y_0 \\ \Delta z = \Delta z_0 \\ \Delta x = \Delta x_0 / \gamma \end{matrix}$ since volume is Lorentz contracted $\Rightarrow \rho = \dfrac{Q}{\Delta x \Delta y_0 \Delta z_0} = \dfrac{Q}{(\Delta x_0/\gamma)\Delta y_0 \Delta z_0} = \gamma\rho_0$

We thus have source 4-vector $\tilde{J}^\mu = \begin{pmatrix} c\rho & \vec{J} \end{pmatrix}$ with $\vec{J} = \rho\vec{u}$

This leads to a covariant form of the continuity Eq.

$\dfrac{\partial\rho}{\partial t} + \vec{\nabla} \bullet \vec{J} = 0$ Using derivative 4 vector $\nabla^\mu = \begin{pmatrix} \dfrac{\partial}{c\partial t} & -\vec{\nabla} \end{pmatrix}$

$\tilde{\nabla} \bullet \tilde{J} = \dfrac{\partial\rho}{\partial t} + \vec{\nabla} \bullet \vec{J} = 0$ continuity equation.

The potentials also form a 4-vector $\tilde{A} = \begin{pmatrix} V/c & \vec{A} \end{pmatrix}$

$\vec{\nabla} \bullet \vec{A} + \dfrac{1}{c^2}\dfrac{\partial V}{\partial t} = 0$ (Lorentz Gauge) written as $\tilde{\nabla} \bullet \tilde{A} = 0$

In Lorentz Gauge we can write potential source equation in covariant form: $\tilde{\nabla} \bullet \tilde{\nabla}\tilde{A} = \mu_0\tilde{J}$ or using $\tilde{\nabla} \bullet \tilde{\nabla} = \dfrac{\partial^2}{c^2\partial t^2} - \nabla^2$

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\begin{pmatrix} \dfrac{V}{c} & \vec{A} \end{pmatrix} = -\mu_0\begin{pmatrix} c\rho & \vec{J} \end{pmatrix}$$

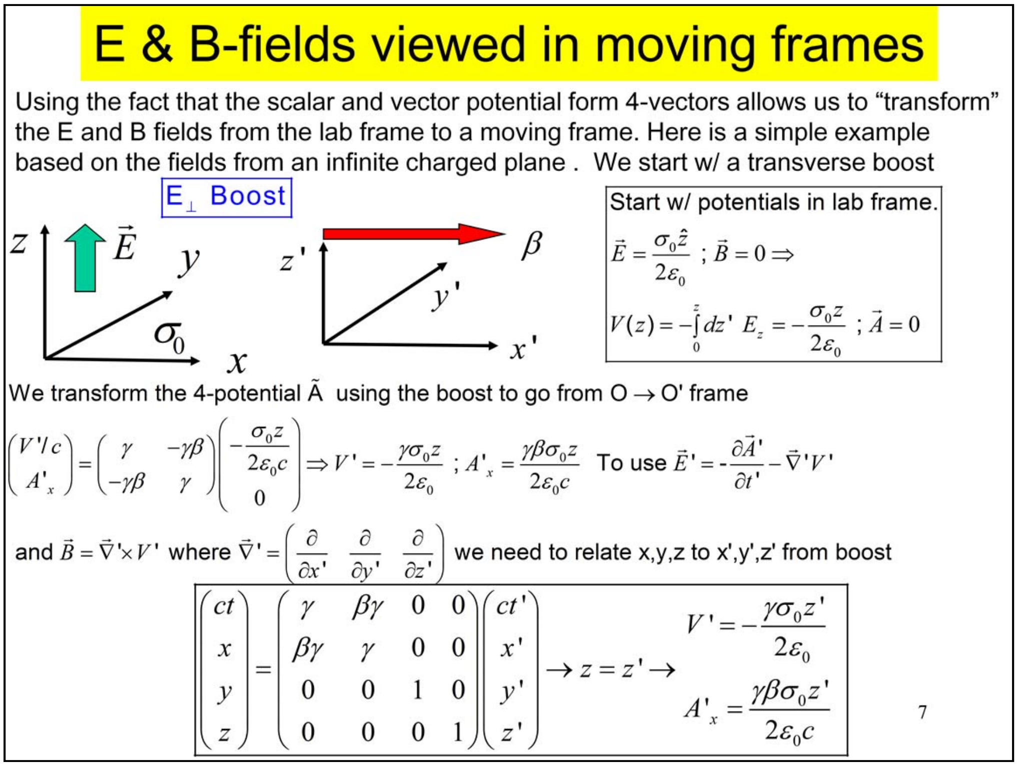
$\left(\nabla^2 - \dfrac{\partial^2}{c^2\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$ with solutions $\vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int_V \dfrac{\vec{J}(\vec{r}\,',t_R)\,d\tau'}{|\vec{r} - \vec{r}\,'|}$

$\left(\nabla^2 - \dfrac{\partial^2}{c^2\partial t^2}\right)V = -\mu_0 c^2\rho = -\dfrac{\rho}{\varepsilon_0} \Rightarrow V(\vec{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \dfrac{\rho(\vec{r}\,',t_R)\,d\tau'}{|\vec{r} - \vec{r}\,'|}$

or $\boxed{\tilde{A}(r,t) = \dfrac{\mu_0}{4\pi}\displaystyle\int_V \dfrac{\tilde{J}(\vec{r}\,',t_R)\,d\tau'}{|\vec{r} - \vec{r}\,'|}}$ where $\tilde{J} = (c\rho, \vec{J})$ & $\tilde{A} = \begin{pmatrix} V/c & \vec{A} \end{pmatrix}$

6

We now turn to how currents, potentials, and fields transform when viewed in different velocity reference frames. Our conclusion will be that current densities, and potentials transform as 4-vectors and fields transform in a more complicated way – as components of 4 by 4 tensor. We are guided by 4 momenta in constructing the 4 current density. The 4 momenta takes a scalar (the particle mass) and multiplies this by the 4 – velocity (eta) to form a 4-vector with a 0 component given by the energy/c and a 1,2,3 component consisting of momentum. For the 4-current we multiply an invariant charge density rho_0 by the 4-velocity eta. Just like tilde-p dot tilde-p = (mc)^2 indicating mass is an invarient, tilde-J dot tilde-j = (c rho_0)^2 indicating rho_0 is an invariant. To interpret rho_0 we go to the non-relativistic limit where tilde-J = (c rho_0, rho_0 vec_u). Since the usual current density is rho times velocity we can think of rho_0 as the charge density in the non-relativistic limit and thus tilde-J is basically a 4-vector of electrodynamic sources: the 0 component is the c rho where rho is the Coulomb source of electric fields, and the 1,2,3 components is the current density or the Biot-Savart source of magnetic fields. Evidently the charge density transforms as rho = gamma rho_0. We can think of it as a “proper” charge density in analogy with a proper time. The proper time is the time interval in the rest frame of a particle while the proper charge density is the charge density in the rest frame of a cloud of charge. The 4-dot product of an interval 4 factor is proportional to the square of the proper time, while the 4-dot product of a 4–current density is proportional to the square of the proper charge density. We can also think of the relationship between the proper charge density and the actual charge density in the following way. Lets assume that the total charge in a cube is invariant. In the rest frame of the cube the volume is dx_0 dy_0 dz_0. We now view the cube in a frame moving along the x-axis. The x length becomes Lorentz contracted (just the height of the mountain to a moving muon) but all other lengths are transverse and unchanged. Thus the volume of the cube gets reduced by a factor of gamma which means the charge density gets increased by a factor of gamma. Hence rho =gamma rho_0. We can use our “newly” discovered 4-current density to build up other elegant 4-vector (i.e. covariant) formulations of the basic equations electrodynamics. For example if we introduce the 4-gradiant as ([partial / c dt], -vec-del) which is a 4-vector with ct derivatives in the 0 component, and - space derivatives in the 1,2,3 component – we can write the continuity equation in an elegant form as tilde-del dot tilde-J = 0. If we introduce a 4-potential as tilde-A = (V/c , vec-A) where V is the scalar potential and vec-A is the vector potential, the Lorentz gauge condition is tilde-del cot tilde-A = 0. In this gauge, the potentials can be found using tilde-del dot tilde-del tilde A = mu_0 tilde J where tilde-del dot tilde-del is the famous d’Alembertian operator which we introduced in our Radiation chapter. Note these 4-vector equations are exactly the same as we have been using (e.g electrodynamics is already “relativity ready”.) The solutions for A and V are given by the retarded form of static expressions involving the rho and J sources. In fact, we can combine these solutions in an elegant looking form as the 4-potential is a volume integral over tilde-J/|vec r-vec r’|. I find this to be a particularly illuminating form since it involves mu_0 which is **exactly** 4 pi times 10^7 and is more of unit conversion rather than a measured quantity. The factors of c in tilde-A and tilde-J change the mu_0 in the vector potential and a 1/epsilon_0 in the scalar potential. In some sense in this formulation , c is the measured quantity which in Maxwell electrodynamics is derived from Coulomb’s constant measured circa1780 using the first torsion balance. Transforming the 4 potential as a 4-vector gives one method for finding the electromagnetic fields in moving frames. For example in homework you will use this technique to compute the electromagnetic fields for a moving charge – a problem which we worked out using the Lienard-Weichert potentials in lecture. This is often an easy way to compute “moving” fields but one often has to transform the coordinate 4- vectors as well as the 4-potentials. Later we show how one can directly transform the fields specified in one frame into another frame without relying on potentials.



Since the potentials are 4-vectors we know how to transform them into moving frames using boosts. We will illustrate this technique for two directions for a constant electrical field. In the first case the field is due to an infinite plane of charge in the x-y with a surface density of sigma_0 which produces constant, uniform electric field in the z-hat direction. Our first step is to convert the electrical field into a potential. We do this by taking negative integral of the E_z along the z-hat direction which insures that our E_z is the negative gradient of the potential. Since there are no currents, we have no vector potential. We next boost the A-tilde 4 potential into the moving prime frame. Since the prime frame is moving along the x-hat direction, only V/c and A_x will mix. We chose negative signs in the boost matrix since a ball rolling along the x-axis in the unprimed frame will be slower in the primed frame. Multiplying our (V/c, A_x) vector by the boost matrix gives us V' and A'_x but unfortunately they are in unprimed (t,x,y,z) coordinates and to convert them into electromagnetic fields we need to take derivatives with respect to primed coordinates. Fortunately the primed potentials only depend on z' and z' = z since the z-coordinate is transverse to the boost velocity. We thus have fairly simple V' and A' expressions in terms of z'.

E transverse boost

$$V' = -\frac{\gamma\sigma_0 z'}{2\varepsilon_0} \; ; \; A' = \frac{\gamma\beta\sigma_0 z'}{2\varepsilon_0 c}\,\hat{x}$$

$$\vec{E}' = -\frac{\partial \vec{A}'}{\partial t'} - \vec{\nabla}' V' \Rightarrow \vec{E}' = -\hat{x}\frac{\partial}{\partial t'}\frac{\gamma\beta\sigma_0 z'}{2\varepsilon_0 c} + \vec{\nabla}'\left\{\frac{\gamma\sigma_0 z'}{2\varepsilon_0}\right\} = \hat{z}\frac{\gamma\sigma_0}{2\varepsilon_0} \rightarrow E_z' = \frac{\gamma\sigma_0}{2\varepsilon_0} = \gamma E_z$$

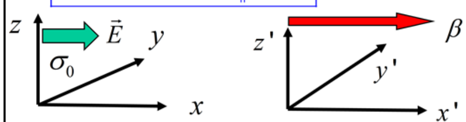
$$\vec{B} = \vec{\nabla}' \times \vec{A}' \rightarrow B_y' = \frac{\partial A_x'}{\partial z'} = \frac{\partial}{\partial z'}\frac{\gamma\beta\sigma_0 z'}{2\varepsilon_0 c} = \frac{\gamma\beta\sigma_0}{2\varepsilon_0 c} = \frac{\gamma v\sigma_0}{2\varepsilon_0 c^2} = \frac{\gamma\beta}{c}E_z = \frac{\gamma v}{c^2}E_z$$

The $\vec{E}$ is easy to understand: $E_z' = \frac{\sigma'}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\frac{Q}{\Delta x'\Delta y'} = \frac{1}{2\varepsilon_0}\frac{Q}{(\Delta x/\gamma)\Delta y} = \gamma\frac{\sigma}{2\varepsilon_0} \rightarrow E_z' = \gamma E_z$

The $\vec{B}$ is understand from Ampere's law applied to current plane $\vec{B} = \frac{\mu_0\left(\vec{K}\times\hat{\eta}\right)}{2}$

$$\vec{K} = -v\sigma\hat{x} = -v(\gamma\sigma_0)\hat{x} \; ; \; \vec{B}' = \frac{\mu_0\left(\vec{K}\times\hat{z}\right)}{2} = \frac{\mu_0(-v\gamma\sigma_0)(\hat{x}\times\hat{z})}{2} = \frac{\mu_0(v\gamma\sigma_0)\hat{y}}{2} \xrightarrow{c^{-2}=\mu_0\varepsilon_0} B_y' = \frac{\gamma v}{c^2}E_z$$

Now consider an $E_{||}$ Boost



Potentials in lab frame.

$$\vec{E} = \frac{\sigma_0\hat{x}}{2\varepsilon_0} \; ; \; \vec{B} = 0 \Rightarrow$$

$$V(z) = -\int_0^x dx'\, E_x = -\frac{\sigma_0 x}{2\varepsilon_0} \; ; \; \vec{A} = 0$$

$$\begin{pmatrix} \frac{V'}{c} \\ A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\sigma_0 x}{2c\varepsilon_0} \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} V' = -\frac{\gamma\sigma_0 x}{2\varepsilon_0} \\ A_x' = \frac{\gamma\beta\sigma_0 x}{2c\varepsilon_0} \end{matrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \rightarrow x = \gamma(vt' + x')$$

8

We compute the gradient of V’ and the time derivative of A_x’ to find the electric field. A’_x doesn’t depend on time and hence only the gradient of V’ is non-zero giving an electrical field in the z-hat direction. After converting sigma_0 to E_z we see the transformed E’_z is just gamma E’_z. We can take the curl of the primed vector potential to find the B-field. Since only the A’_x component is non-zero and it only depends on z’ the B-field must lie in (negative) y-hat direction. We can write the sigma_0 piece using E_z and find that it is proportional to epsilon_0 mu_0 E_z or E_z/c^2 as well as gamma times the velocity. We can understand the forms of the fields in the primed frame. The electrical field in the prime frame is again due to the charge density in the x-y plane but the charge density is larger by a factor of gamma due to the Lorentz contraction along the boost (or x-hat) direction. This causes the E’_z to increase in the primed frame. We get a non-zero magnetic field in the primed frame since in this frame the surface charge is moving in the – x-hat direction creating a surface current in the negative x-hat direction given by - sigma’ v. Again sigma’ is gamma sigma. Since the primed electrical field is constant we can compute the B-field from the surface charge using Ampere’s law in our current plane form. Here eta is the normal to the point which is z-hat. In the primed frame , the surface current is in the –x-hat direction and is of the sigma times v form. Sigma is increased from sigma_o by a factor of gamma as was the case for the E-field. We find the primed magnetic field is in the y-hat direction and is the same answer we got from curl of the potential.

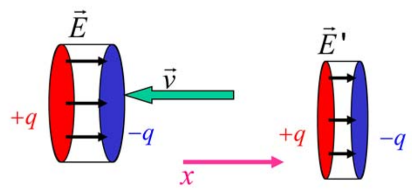
We now consider the same problem but now put the E-field along the x-direction in the un-primed frame. The scalar and vector potential are functions of x which now depends on both x’ and t’ .

The parallel boost continues…

$$V' = -\frac{\gamma\sigma_0 x}{2\varepsilon_0} = -\frac{\gamma^2\sigma_0}{2\varepsilon_0}\left(vt'+x'\right) \; ; \; A_x' = \frac{\gamma^2\beta\sigma_0}{2c\varepsilon_0}\left(vt'+x'\right) \; ; \; \vec{B}' = \vec{\nabla}'\times\vec{A}' = 0 \; ; \; \vec{E}' = -\frac{\partial\vec{A}}{\partial t'} - \vec{\nabla}'V'$$

$$\frac{\partial\vec{A}}{\partial t'} = \hat{x}\,\frac{\gamma^2\beta\sigma_0}{2c\varepsilon_0}\frac{\partial\left(vt'+x'\right)}{\partial t'} = \frac{\gamma^2 v\beta\sigma_0}{2c\varepsilon_0}\hat{x} = \frac{\gamma^2\beta^2\sigma_0}{2\varepsilon_0} \; ; \; \vec{\nabla}'V' = -\hat{x}\,\frac{\gamma^2\sigma_0}{2\varepsilon_0}\frac{\partial\left(vt'+x'\right)}{\partial x'} = -\hat{x}\,\frac{\gamma^2\sigma_0}{2\varepsilon_0}$$

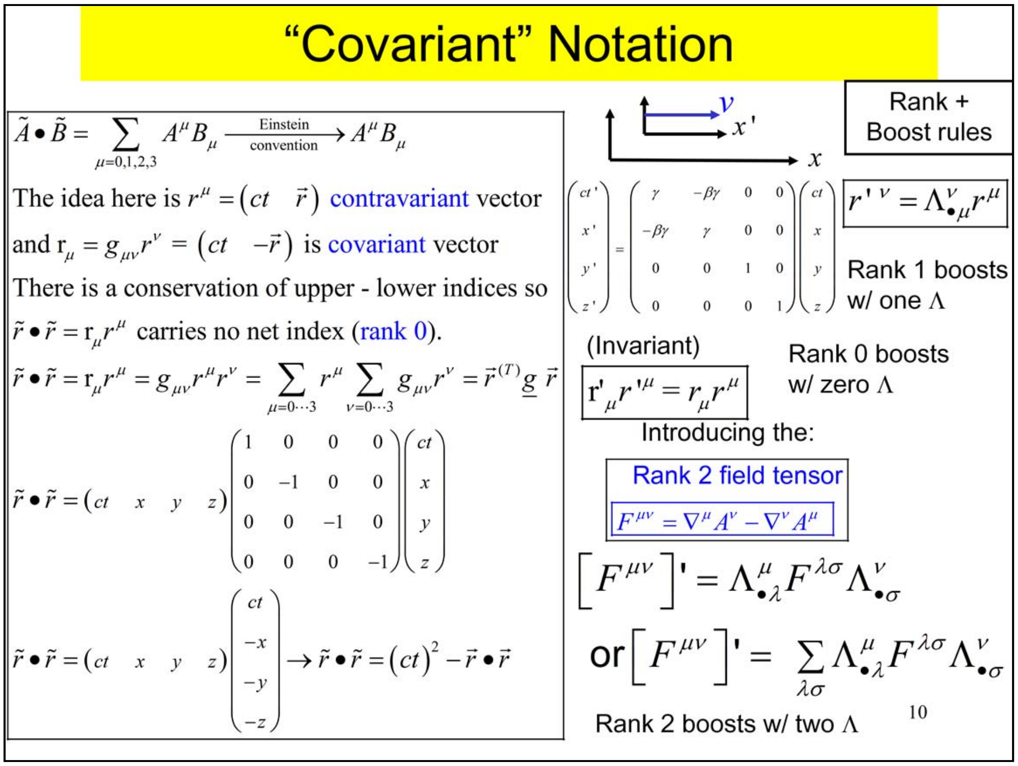
$$\vec{E}' = -\frac{\partial\vec{A}}{\partial t'} - \vec{\nabla}'V' \Rightarrow \vec{E}' = \frac{\sigma_0}{2\varepsilon_0}\hat{x}\left[\gamma^2 - \frac{\gamma^2\beta v}{c}\right] = \frac{\sigma_0}{2\varepsilon_0}\hat{x}\left[\gamma^2 - \beta^2\gamma^2\right] = \frac{\sigma_0}{2\varepsilon_0}\hat{x} \rightarrow \boxed{E_x' = E_x}$$



The E|| boost is equivalent to boosting a capacitor along gap direction but the field doesn't depend on the gap so $E_{||}' = E_{||}'$

9

The primed magnetic field is zero since A'_x only depends on x' and t' and thus its curl vanishes. Since now A' and E' depend on t' as well as x', the E' calculation will involve the time derivative of A' and the x' derivative of V'. Interestingly enough these two derivatives interact to cancel all of the boost effects leaving a E_x' which equals the E_x in the unprimed frame. The cancellation of all relativistic effects is easy to understand intuitively. If we create the field from a capacitor, the charged planes are transverse to the boost direction and thus the surface charge density is unchanged since there is no "transverse" Lorentz contraction. The gap between the plates is Lorentz contracted but the field does not depend on the gap length and hence the electric field is not affected by the boost.



There is a more systematic and direct way of boosting fields from frame to frame but it requires some important mathematical overhead. We review and further describe covariant notation which we first discussed in our first relativity chapter. We typically use greek indices which range from 0 for time-like component to 3 . The relativistic dot product involves a sum over mu. We use the Einstein convention which implies that repeated indices are summed— in this case summed from 0 to three. We see that the number of upper indices and lower indices is "conserved". The difference in this number is called the tensor rank – which is essentially the number of dimensions for the object. The dot product have one raised index and one lowered index or zero net indices implying a it is "rank zero) and has only one component and is a scalar and doesn't change under rotations and boosts. Essentially all indices are summed over which leaves no "free" indices. The 4-vectors we have been using are actually the contravariant (upper index) components. The covariant forms (lower indices) can be obtained by multiplying by the contravariant 4 –vector by the 4 by 4 metric matrix which reverses the spatial components. The rank zero dot product is constructed from the product of contravariant and covariant components.

The rank of an object tells how it transforms under a boost. The (contravariant) 4-vector boosts by multiplication by the usual 4 by 4 Lorentz boost matrix. In order to conserve rank we write the boost matrix with one upper and one lower index. Thus a rank one object (4-vector) requires one boost matrix. The dot product with one upper and one lower index has rank 0 and is invariant under boosts thus no boost matrix is required. The field tensor (which we will discuss later) is constructed from products of the 4-gradiant and 4-potential. Neither index is summed so there are two free indices meaning the field tensor is rank two. We will show that it boosts in a way proportional to two boost matrices. We write the transformation in repeated indices summed (ala Einstein) and explicit sum notation. We next show that the field tensor "contains" the E and B fields which evidently change in a way requiring two boost matrices. Interestingly enough our potential method also required two boosts: one boost for the 4-potential, and one boost for the coordinates.

Covariant fields from 4-potentials

$\tilde{A}$ and $\tilde{\partial}$ are 4-vectors. Can we write $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$; $\vec{B} = \vec{\nabla} \times \vec{A}$ covariantly?

Consider the derivative matrix $F^{\mu\nu} = \nabla^{\mu} A^{\nu} - \nabla^{\nu} A^{\mu}$ Each $F^{\mu\nu}$ corresponds to a field component.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

These components use our metric and thus differ from Griffiths

$$g_{\mu\nu}^{(\text{ours})} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \neq g_{\mu\nu}^{(\text{Griffiths})} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\vec{E}$ & $\vec{B}$ are not 4-vectors but appear in $F^{\mu\nu}$ built from 4-vector "products" $\tilde{A}$ & $\tilde{\nabla}$

Verify some $F^{\mu\nu} = \nabla^{\mu} A^{\nu} - \nabla^{\nu} A^{\mu}$ component

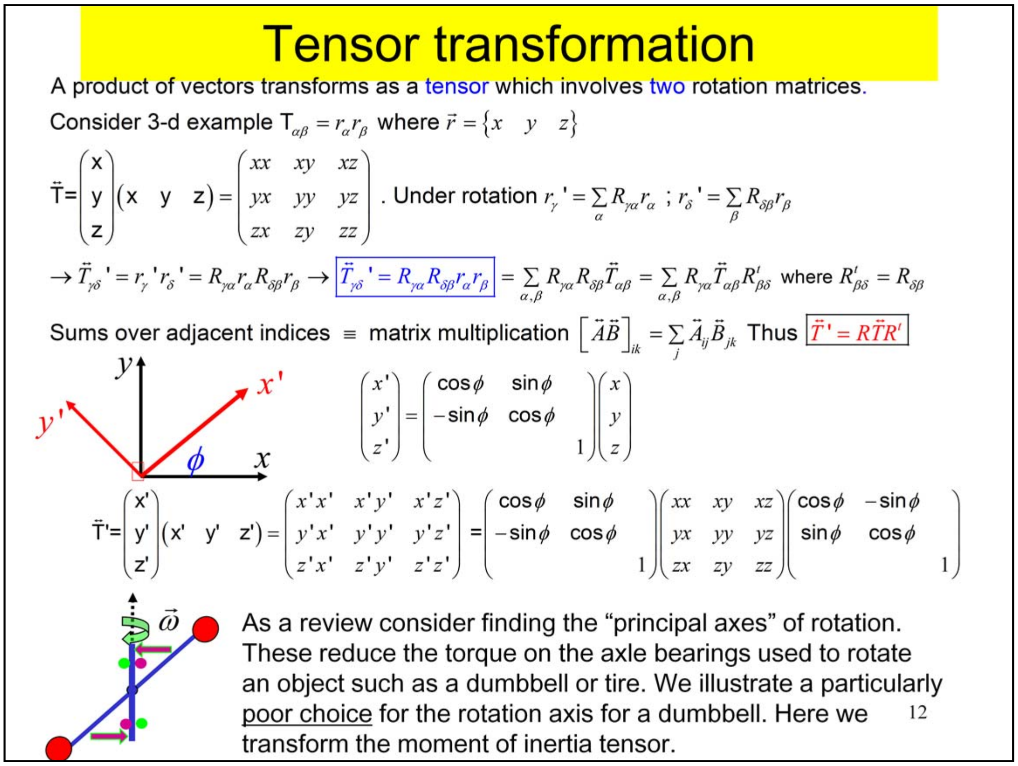
Use $A^{\mu} = \begin{pmatrix} \frac{V}{c} & A_x & A_y & A_z \end{pmatrix}$; and $\nabla^{\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix}$

$$-\frac{E_x}{c} = F^{01} = \nabla^0 A^1 - \nabla^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} - -\frac{\partial}{\partial x}\frac{V}{c} \rightarrow E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}$$

$$B_z = F^{21} = \nabla^2 A^1 - \nabla^1 A^2 = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} \rightarrow B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

11

The first step in field transformations is to write an expression for the field components in terms of covariant derivatives of the 4-potentials. We do this to learn how field components transform (i.e. as second rank tensors). We do this by writing the F^mu nu tensor which is given as an asymmetric derivative of partial A_nu/partial x_mu. The various F^mu nu components can be written as various E-field and B-field components using the classical forms of B is the curl of A and E is the negative time derivative of A – the gradient of V. We illustrate this for F^01 = -Ex/c and F21 = +Bz. The particular form of the F^mu nu tensor depends on one's metric. My signs correspond to the usual High Energy Physics metric which we have been using throughout the SR chapters rather than Griffiths' rather unusual metric. The key point is the field components are "products" of the (partial /partial x_mu) and A^nu contravariant 4 vectors. The "products" of two vectors is called a tensor which rotates (or boost transforms) in a particular way involving the product of two rotation matrices.



Here is a tensor which is slightly simpler than the moment of inertia tensor using in mechanics. Its components are just T_(alpha beta)=r_alpha r_beta. If we were to reconstruct the tensor in a rotated coordinate (primed) system, we would rotate (x y z) into (x' y' z') and then rebuild the matrix from say T_11= xx to x'x' where now x' and y' are functions of x y z. We do this formally in the slide and find that while a rotated column vector is pre-multiplication of the vector by a rotation matrix R, a rotated tensor involves a pre-multiplication by R and a post-multiplication by R^(transpose). The transformation rule using Einstein convention (in blue) is essentially identical to the F'^munu transformation (apart from contravariance). Thus vectors require one rotation multiplication while tensors require two. For the relativistic 4 by 4 field tensor that we will consider shortly we will need a multiplications by a pre-boost and a post-boost to transform the field tensor to a moving frame.

Transforming fields by "rotating" $F^{\mu\nu}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \qquad \Lambda^{\sigma}_{\bullet\lambda} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

We use a double rotation rule to transform F tensor

$$\left[F^{\mu\nu}\right]' = \Lambda^{\mu}_{\bullet\lambda} F^{\lambda\sigma} \Lambda^{\nu}_{\bullet\sigma}$$

Prove rotation rule

$$\left[F^{\mu\nu}\right]' = \nabla^{\mu}{}' A^{\nu}{}' - \nabla^{\nu}{}' A^{\mu}{}' = \left(\Lambda^{\mu}_{\bullet\lambda}\nabla^{\lambda}\right)\left(\Lambda^{\nu}_{\bullet\sigma}A^{\sigma}\right) - \left(\Lambda^{\nu}_{\bullet\sigma}\nabla^{\sigma}\right)\left(\Lambda^{\mu}_{\bullet\lambda}A^{\lambda}\right)$$

$$\left[F^{\mu\nu}\right]' = \Lambda^{\mu}_{\bullet\lambda}\Lambda^{\nu}_{\bullet\sigma}\left[\nabla^{\lambda}A^{\sigma} - \nabla^{\sigma}A^{\lambda}\right] = \Lambda^{\mu}_{\bullet\lambda}\Lambda^{\nu}_{\bullet\sigma}F^{\lambda\sigma}$$

Illustrate a few rotations

$$\left[F^{01}\right]' = \Lambda^{0}_{\bullet\lambda}\Lambda^{1}_{\bullet\sigma}F^{\lambda\sigma} = \Lambda^{0}_{\bullet 0}\Lambda^{1}_{\bullet 0}\cancel{F}^{00} + \Lambda^{0}_{\bullet 1}\Lambda^{1}_{\bullet 0}F^{10} + \Lambda^{0}_{\bullet 0}\Lambda^{1}_{\bullet 1}F^{01} + \Lambda^{0}_{\bullet 1}\Lambda^{1}_{\bullet 1}\cancel{F}^{11}$$

$$\left(\frac{-E_x{}'}{c}\right) = \left(-\gamma\beta\right)^2\left(\frac{E_x}{c}\right) + \gamma^2\left(\frac{-E_x}{c}\right) \rightarrow E_x{}' = \left[\gamma^2 - \left(\gamma\beta\right)^2\right]E_x = \frac{1-\beta^2}{1-\beta^2}E_x = E_x$$

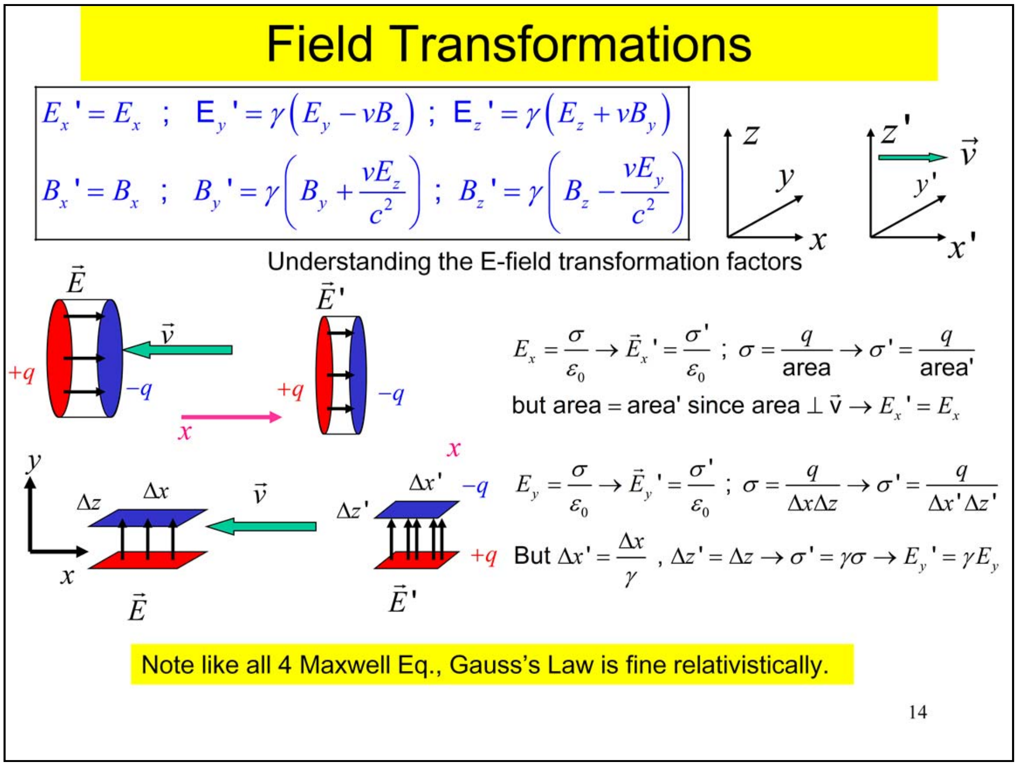
$$B_y{}' = \left[F^{13}\right]' = \Lambda^{1}_{\bullet\lambda}\Lambda^{3}_{\bullet\sigma}F^{\lambda\sigma} = \sum_{\lambda}\Lambda^{1}_{\bullet\lambda}F^{\lambda 3} = \Lambda^{1}_{\bullet 0}F^{03} + \Lambda^{1}_{\bullet 1}F^{13}$$

$$B_y{}' = -\gamma\beta\left(-\frac{E_z}{c}\right) + \gamma B_y = \gamma\left(B_y + \frac{vE_z}{c^2}\right)$$

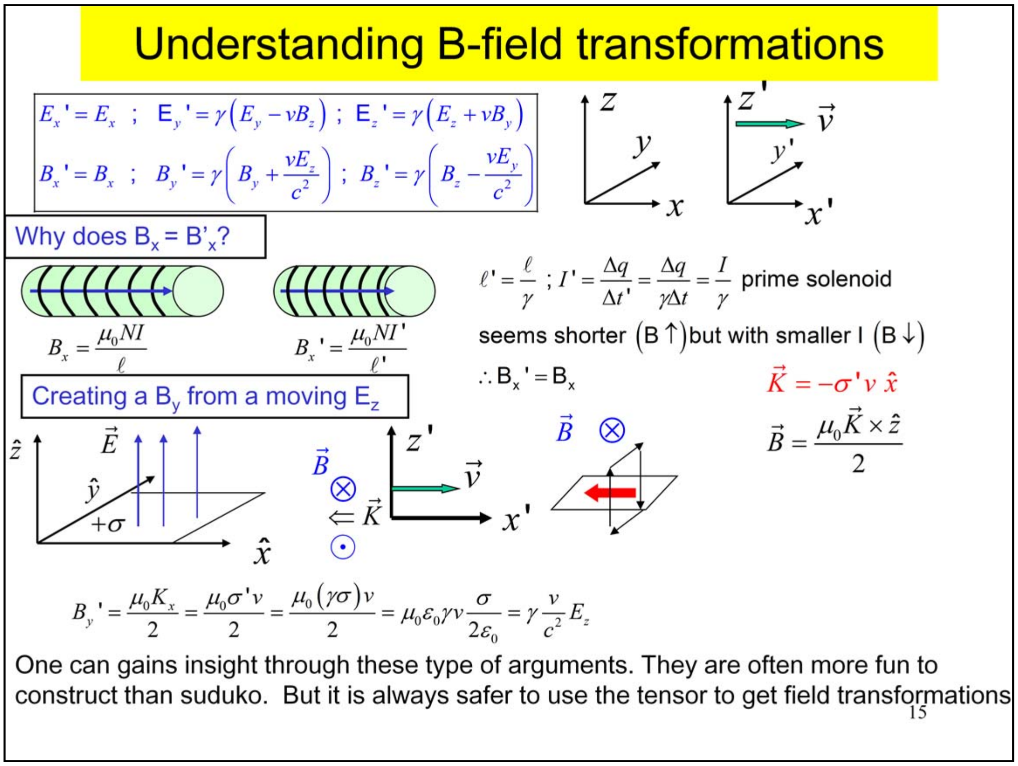
13

We follow the cue on how to rotate the 3-d I tensor and write a tensor boost rule involving two boost matrices. Lets show that this tensor boost works. We write the transformed field tensor according to our assumed form (in red). The field tensor consists of two terms. In the first term, we write the transformed Del^mu as multiplication by the boost matrix using a dummy index lambda and the transformed A^nu as multiplication by a boost matrix using a dummy index sigma. In the second term we switch the dummies and write the transformed Del^nu using the dummy index sigma and the transformed A^mu using a dummy index lambda. The switching allows us to factor out the two Lambda factors to get our "double rotation" matrix version of the transformation.

We now illustrate "rotation" of the field tensor. We are just boosting to a primed frame which is moving with a velocity beta c x-hat with respect to the unprimed frame where the field components are given. There are several things to note about the boost which I write as a 4 by 4 matrix. We note that only x and t are affected by the boost parameters and we use - beta gamma. This is because the velocity of a ball moving along the x axis in the primed frame has a smaller velocity in the unprimed frame. We also note that the boost matrix is very sparse – most components are zero. I thus find it easier to construct the double rotation by components rather than by explicit matrix multiplication. We illustrate a few field boosts. For example lets say we want to know the Ex field in the moving (prime) frame. This appears twice in the field tensor – we will use F'^01 which is –Ex'/c. We next write F'^01 in terms of a F^lambda sigma times boost parameters. From the double boost form we know the mu boost index is 0 and the nu boost index 1 and lambda and sigma run over 0,1,2,3. Now only Lambda^0_1 and Lambda^0_0 are non-zero for mu=0. Similarly only Lambda^0_1 and Lambda^0_0 are non zero for nu=1. Hence we only have four terms in the double boost. But F^00 = F^11 =0 meaning only two of the 4 terms survive. We write out these two surviving terms which involves the product of the two diagonal and two off diagonal boost parameters and find Ex' = Ex. Hence E|| is unchanged by the boost. We next illustrate a magnetic field boost for By' which we write as F'^13 . This term will involve mu =1 and nu = 3. Now nu = 3 only connects with Lambda^3_3 =1 while mu = 1 connects with Lambda^1_0 and Lambda^1_1 which is – gamma beta and gamma respectively. The field tensor component that multiplies (–gamma beta) is F^03 = -Ez/c while the component that multiplies gamma is F^13 which is By. Hence By' involves a transverse component of B as well as E.

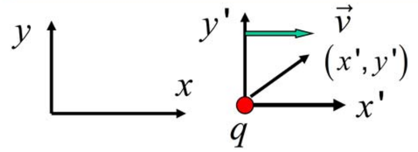


Here is the complete set of field transformations for this x axis boost. Electric and magnetic fields aligned parallel to the boost axis are left invariant, whereas the transverse field components in the primed frame involve both transverse electric and magnetic fields in the unprimed frame. It is generally easy (and fun) to construct physical arguments for these six field transformations. For example consider Ex = E'x where we generate Ex using two infinite sheets of charge in the y-z plane. The primed frame sees these plates moving to the left with velocity –v. As before we assume the charge is invariant under the boost and the transverse area is invariant as well since perpendicular dimensions are unchanged. Hence the surface charge density sigma is unchanged. The separation of the two plates is Lorentz contracted and thus down by a factor of gamma. But the Ex field only depends on sigma and not the plate separation so E'x = Ex which agrees with our tensor transformation rules. But the field transformation rules say that the transverse E components are changed by a boost. We can check the E'y transformation rules for the case where the unprimed frame has no magnetic field but with an electrical field, due to a capacitor with separated plates along the y axis. In this case sigma' is increased by a factor of gamma since sigma = q / (delta x delta y) and delta x is Lorentz contracted according to delta x' = delta x/gamma. Since the E'y field is proportional to sigma, E'y = gamma Ey which checks the electrical part of one (i.e. y) of the Eperp transformations. An alternative way of viewing this is the E-field is proportional to the density of the electrical field lines. The field lines are closer together in the x-direction in the boosted frame and hence their density increases. The Ez field will also increase by a factor of gamma by the same capacitor , or field line density argument. We note that we used Gauss's law to construct our field transformation argument so evidently Gauss's law still works even at high velocities. But again, all of Maxwell's equations are "relativistic-ready".



We can also check some of the magnetic transformations. For example, the parallel components of B are unchanged by the boost. We can check this using the magnetic field of a solenoid which you recall from Physics 212 is proportional to the current times the windings per meter or the number of windings N times the current I divided by the solenoid length L. In the primed frame L’ = L/gamma since the solenoid is Lorentz contracted. The current is also decreased by i’ = i/gamma. The current is the charge passing a point divided by the time interval that the current flows. The charge is unchanged but the time interval delta T is time dilated to delta T’ = gamma delta T and hence i’ = i/gamma. Since the B-field is proportional N i /L and i and L both are reduced by a factor of gamma in the prime frame and the number of loops is unchanged the parallel B-field is unchanged. It is difficult to get the By or Bz transformation using this argument since the circular windings will turn into elliptical windings under a Lorentz boost. We can also partially check the E-field contribution to the magnetic field transformation. We consider a uniform plane of charge in the x-y plane of the unprimed frame. We want to find the magnetic field for a point in the +z direction in a primed frame moving with a velocity -v x-hat wrt the prime frame. An observer in the prime frame sees the (sigma’) surface charge moving in the –x-hat direction and hence a surface current of vec-K’ =-(sigma’) v x-hat. We can find the B-field for z > 0 using K cross eta and find By = mu_0 K’/2. Inserting this surface current, and writing sigma’ = gamma sigma from our moving capacitor problem we get By in terms of sigma in the unprimed frame. We can then write sigma in terms of Ez in the unprimed frame to get the electrical contribution to the transformed B-field. Griffiths works out additional checks of the transformation. Its fun to work these out, but ultimately the tensor transformation rules are the safest way of relating the electromagnetic fields in moving frames.

Revisiting the Lienard- Weichert potential

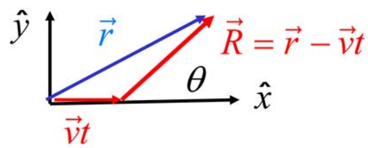


$$\begin{pmatrix} V/c \\ A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & \\ \gamma\beta & \gamma & \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V'/c \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} V = \gamma V' \\ A_x = \gamma\beta \dfrac{V'}{c} \end{matrix}$$

Suggesting $\vec{A}(\vec{r},t) = \dfrac{\vec{v}}{c^2} V(\vec{r},t)$

$$V'(x',y') = \frac{q}{4\pi\varepsilon_0\sqrt{x'^2+y'^2}} \rightarrow V(x',y') \frac{\gamma q}{4\pi\varepsilon_0\sqrt{x'^2+y'^2}} \; ; \; \begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \end{pmatrix}$$

$$\begin{matrix} t' = \gamma t - \gamma\beta x / c \\ x' = \gamma(x - \beta ct) \\ y' = y \end{matrix} \quad \rightarrow V(x,y,t) \frac{\gamma q}{4\pi\varepsilon_0\sqrt{\gamma^2(x-vt)^2 + y^2}}$$



$$x - vt = R\cos\theta \; ; \; y = R\sin\theta \; ; \; \gamma^2(x-vt)^2 + y^2 = \gamma^2 R^2 \cos^2\theta + R^2 \sin^2\theta$$

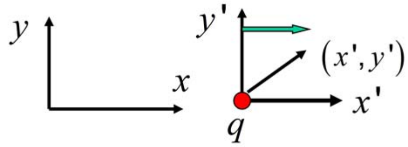
$$= \gamma^2 R^2 \left(\cos^2\theta + \frac{\sin^2\theta}{\gamma^2} \right) = \gamma^2 R^2 \left(\cos^2\theta + (1-\beta^2)\sin^2\theta \right) = \gamma^2 R^2 \left(1 - \beta^2 \sin^2\theta \right)$$

$$\rightarrow V(\vec{r},t) = \frac{\gamma q}{4\pi\varepsilon_0 R\gamma\sqrt{1-\beta^2\sin^2\theta}} = \frac{q}{4\pi\varepsilon_0 R\sqrt{1-\beta^2\sin^2\theta}}$$ same as Griffiths G10.14

16

We revisit the Lienard- Weichert potential for a charge at the origin moving with a uniform velocity. As you recall from Homework #7 this is a rather complicated calculation. We begin with the Coulomb potential in a frame moving with the charge q and boost it to the lab frame. We use positive signs in the boost since a ball rolling along the x' axis will move faster in the unprimed frame. We find that V picks up a factor of gamma and a vector potential is created in the lab frame which is proportional to the scalar potential with exactly the L-W form. We need to write V(x,y) rather than our form with V(x',y') and thus need to boost the coordinates from x,y to x',y'. We end up with a fairly simple form for the potential in the lab frame. It is illuminating to write the coordinate in terms of the R-vector which is the vector from the observation point to the present position of the charge (as opposed to the retarded position). We get the same expression as that obtained using retarded potentials but in a much more straightforward way using Relativity. We don't need to invoke the subtle argument concerning "retarded" charge, and the morass of algebra required to write the result in terms of sin(theta). We rapidly get the surprising conclusion that the potential depends on **present** rather retarded displacement or R = |r-vec-v-vec t|.

The E-field of a moving charge redux



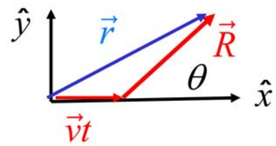
Griffiths 10.68

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V = \frac{q\hat{R}}{4\pi\varepsilon_0 R^2}\frac{1-\beta^2}{\left(1-\beta^2 \sin^2\theta\right)^{3/2}}$$

where $\vec{R} = \vec{r} - \vec{v}t$ and $\beta = v/c$

$$\vec{E}' = \frac{q\left(x'\hat{x} + y'\hat{y} + z'\hat{z}\right)}{4\pi\varepsilon_0\left(x'^2 + y'^2 + z'^2\right)^{3/2}} \; ; \; \vec{R} = \begin{pmatrix} x - vt & y & z \end{pmatrix}$$

$$x' = \gamma\left(x - vt\right) = \gamma R_x \; ; \; y' = y = R_y \; ; \; z' = z = R_z$$



$$\vec{E}' = \frac{q\left(\gamma R_x\hat{x} + R_y\hat{y} + R_z\hat{z}\right)}{4\pi\varepsilon_0\left[\left(\gamma R_x\right)^2 + R_y^2 + R_z^2\right]^{3/2}} \; ; \; E_{\parallel} = E_{\parallel}' \text{ and } E_{\perp} = \gamma E_{\perp}' \;\Rightarrow \vec{E} = \begin{pmatrix} \vec{E}_x' & \gamma\vec{E}_y' & \gamma\vec{E}_z' \end{pmatrix}$$

Thus $$\vec{E} = \frac{q\left(\gamma R_x\hat{x} + \gamma R_y\hat{y} + \gamma R_z\hat{z}\right)}{4\pi\varepsilon_0\left[\left(\gamma R_x\right)^2 + R_y^2 + R_z^2\right]^{3/2}} = \frac{\gamma q\vec{R}}{4\pi\varepsilon_0\left[\left(\gamma R_x\right)^2 + R_y^2 + R_z^2\right]^{3/2}} = \frac{\gamma q\vec{R}/R^3}{4\pi\varepsilon_0\left[\left(\gamma\cos\theta\right)^2 + \sin^2\theta\right]^{3/2}}$$

$$\vec{E} = \frac{q\hat{R}}{4\pi\varepsilon_0 R^2}\frac{\gamma^{-2}}{\left[\cos^2\theta + \gamma^{-2}\sin^2\theta\right]^{3/2}} \Rightarrow \vec{E} = \frac{q\hat{R}}{4\pi\varepsilon_0 R^2}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} \text{ as before but } \approx \frac{1}{10} \text{ effort!}$$

17

On this slide we obtain the electrical field for a moving point charge with a constant velocity using relativistic methods. We worked out the “one dimensional” case where the charge moves in line with the observer using the Lienard-Weichert potentials in our radiation chapter and quoted the 3-d result worked out in your text. One surprising result of this calculation was that the electrical field can be written as a simple function of the vec-R which the displacement vector from the actual displacement of the observer relative to the charge and **not** the retarded position of the charge as one would have expected in a retarded potential problem. You verified this form in homework and as you recall it required several remarkable algebraic cancellations. Here we work out this problem by constructing the E-field in the primed frame which is the charge rest frame and transforming the perpendicular and parallel electrical field components. We also have to transform the position of the charge in the unprimed frame to its position in the primed frame where the primed field is calculated. The primed frame electrical field is just given by Coulomb’s law and there is no magnetic field in this frame. The displacement vector in unprimed coordinates is time dependent since the charge is retreating from the observer along the x-axis. We recycle the boost from x,y,z to x’,y’,z’. We next write the electrical field E’ in terms of the unprimed relative coordinates vec-R. The primed field does not point along vec-R since the x-hat component gets a factor of gamma relative to the y-hat and z-hat components. However the y and z components are perpendicular components and pick up a factor of gamma from the field transformation and hence the unprimed field is directed along vec-R or the un-retarded relative position. After some simple algebraic manipulation we recover the same form we had using retarded potential methods but in a much more straightforward way. Again, the point is that the relativistic treatment has the same physics as the classical method (e.g. classical electrodynamics is already “relativistic-ready”).

Field Invariants

Write invariant $F^{\mu\nu}F_{\mu\nu}$ in terms of 4×4 matrices: $\underline{F}$ and $\underset{\sim}{F}$

$$F^{\mu\nu} = \left[\underline{F}\right]_{\mu\nu} \text{ and } F_{\mu\nu} = \left[\underset{\sim}{F}\right]_{\mu\nu} = \left[\underset{\sim}{F}^{(t)}\right]_{\nu\mu}$$

$$\rightarrow F^{\mu\nu}F_{\mu\nu} = \sum_{\mu}\sum_{\nu}\left[\underline{F}\right]_{\mu\nu}\left[\underset{\sim}{F}^{(t)}\right]_{\nu\mu} = \sum_{\mu}\left[\underline{F}\,\underset{\sim}{F}^{(t)}\right]_{\mu\mu} = Tr\left\{\underline{F}\,\underset{\sim}{F}^{(t)}\right\}$$

Construct $\underset{\sim}{F}$

$F_{\mu\nu} = \sum_{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}$ where $g_{\mu\alpha} = diag(1 \;\; -1 \;\; -1 \;\; -1)$. This means $F_{\mu\nu} = F^{\mu\nu}$ if μ and ν are both either time-like or space-like and $F_{\mu\nu} = -F^{\mu\nu}$ if μ is time-like , ν is space-like or vice versa. Thus means $\vec{E} \rightarrow -\vec{E}$ and $\vec{B} \rightarrow \vec{B}$ as $\underline{F} \rightarrow \underset{\sim}{F}$.

Since $F_{\mu\nu} = -F_{\nu\mu}$, $\underset{\sim}{F} \rightarrow -\underset{\sim}{F}^{(t)}$ thus $\vec{E} \rightarrow \vec{E}$ and $\vec{B} \rightarrow -\vec{B}$ as $\underline{F} \rightarrow \left(\tilde{F}\right)^t$

$$\underline{F} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \;\&\; \underset{\sim}{F}^{(t)} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ +E_x/c & 0 & +B_z & -B_y \\ +E_y/c & -B_z & 0 & +B_x \\ +E_z/c & +B_y & -B_x & 0 \end{pmatrix}$$

18

We can use the field tensor to construct two invariants which have the same value in any boost frame. The idea is to construct quantities with rank zero which means the same number of contravariant indices as covariant indices. The first quantity is F^mu nu F_mu nu. This is a rank zero object with all indices summed over so it is a Lorentz scalar. This is the two index version of the dot product invariant which is A^mu A_mu. I find it easiest to evaluate using matrix techniques. I will write F^mu nu as the mu nu component of a 4 by four matrix F underscore bar and F_mu nu as the mu nu component of a matrix F underscore tilde. If we take the transpose of F_tilde, our nu index is lined up in the right order for matrix multiplication under a summation of nu. The summation over mu sums the diagonal F_bar F_tilde^(t) components which is the trace of F_bar F_tilde^(t) . We know the matrix form of F_bar and need the matrix form of F_tilde^(t).

To switch from contravariant to covariant forms requires two metrics. Our metric is that time-like components get +1 and space-like components get -1. This means one changes the sign of any component with mu=0 and nu = 1,2,3 or vice versa. These are the positions of the electric field so E changes sign going from F_bar to F_tilde The magnetic fields appear in purely space-like position so no change sign is necessary. Since F_tilde is an antisymmetric tensor, its transverse will flip all signs. With this additional sign flip, we negate the signs of magnetic fields and keep the sign of all electric fields in going from F to F_tilde^(t) .

Field Invariants (continued)

$\underline{F} \equiv F^{\mu\nu}$ $\qquad$ $\tilde{F}^{(t)}$ $\left[\vec{E} \to \vec{E} \;\&\; \vec{B} \to -\vec{B}\right]$

$$\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ +E_x/c & 0 & -B_z & B_y \\ +E_y/c & B_z & 0 & -B_x \\ +E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ +E_x/c & 0 & +B_z & -B_y \\ +E_y/c & -B_z & 0 & +B_x \\ +E_z/c & +B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} M_{00} & * & * & \\ * & M_{11} & * & \\ & & M_{22} & \\ & & & M_{33} \end{pmatrix}$$

$$Tr\left\{\underline{F}\tilde{F}^{(t)}\right\} = M_{00} + M_{11} + M_{22} + M_{33} \;;$$

$$M_{00} = \langle 0 \quad -E_x/c \quad -E_y/c \quad -E_z/c \rangle \bullet \langle 0 \quad +E_x/c \quad +E_y/c \quad +E_z/c \rangle = -\left|\vec{E}\right|^2 / c^2$$

$$M_{11} = \langle -E_x/c \quad 0 \quad -B_z \quad B_y \rangle \bullet \langle +E_x/c \quad 0 \quad -B_z \quad B_y \rangle = B_z^2 + B_y^2 - \left(E_x/c\right)^2$$

$$M_{22} = \langle -E_y/c \quad B_z \quad 0 \quad -B_x \rangle \bullet \langle +E_y/c \quad B_z \quad 0 \quad -B_x \rangle = B_z^2 + B_x^2 - \left(E_y/c\right)^2$$

$$M_{33} = \langle -E_z/c \quad -B_y \quad B_x \quad 0 \rangle \bullet \langle +E_z/c \quad -B_y \quad B_x \quad 0 \rangle = B_y^2 + B_x^2 - \left(E_z/c\right)^2$$

$$\to F^{\mu\nu}F_{\mu\nu} = Tr\left\{F\tilde{F}^{(t)}\right\} = M_{00} + M_{11} + M_{22} + M_{33} = 2\vec{B} \bullet \vec{B} - 2\vec{E} \bullet \vec{E}/c^2$$

Another invariant is $\varepsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} = \frac{8}{c} \vec{E} \bullet \vec{B}$

19

Our next task is to compute the trace of F_bar F_tilde^{t} by matrix multiplication. The trace only involves the 4 diagonal elements. We construct these 4 matrix products and add the results to get the trace. We find that the trace is just twice the square of the B field minus twice the square of the E field: so B^2 –E^2/c^2 is a field invariant with the same value in any reference frame. The other invariant is written in terms of the 4 dimensional version of the Levi-Civita symbol. This symbol is a quantity – which you may have seen in determinants – which is +1 for circular permutations of (a b c d) = (0 1 2 3) and -1 for circular permutations of (1 0 2 3). Again since there are 4 contravariant indices and 4 covariant indices and no free indices, this object is a rank zero scalar or a relativistic invariant. I will just quote the result that this invariant is proportional to E dot B. One can also show the this invariant is proportional to the square of the determinant of F^mu nu.

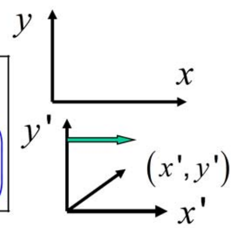
Illustrating Field Invariants

A photon has $\vec{B} = \frac{\hat{k} \times \vec{E}}{c}$ in all frames thus $\vec{B} \bullet \vec{E} = 0$ and $\vec{B} \bullet \vec{B} - \frac{\vec{E} \bullet \vec{E}}{c^2} = 0$ in all frames

Field Transformations

$$E_x' = E_x \;\; ; \;\; E_y' = \gamma\left(E_y - vB_z\right) \; ; \; E_z' = \gamma\left(E_z + vB_y\right)$$

$$B_x' = B_x \;\; ; \;\; B_y' = \gamma\left(B_y + \frac{vE_z}{c^2}\right) \; ; \; B_z' = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$$



if $\vec{E} = E_0\hat{z}$ and $\vec{B} = 0$ then $\vec{E}' = \gamma E_0 \hat{z}$, $\vec{B}' = \frac{\gamma\beta E_0}{c}\hat{y}$

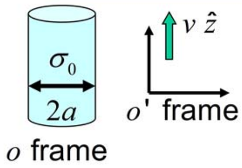
$$\vec{B} \bullet \vec{E} = 0 \; ; \; \vec{B}' \bullet \vec{E}' = \left[\frac{\gamma\beta E_0}{c}\hat{y}\right] \bullet \left[\gamma E_0 \hat{z}\right] = 0$$

$$\vec{B} \bullet \vec{B} - \frac{\vec{E} \bullet \vec{E}}{c^2} = -\frac{E_0^2}{c^2} \; ; \; \vec{B}' \bullet \vec{B}' - \frac{\vec{E}' \bullet \vec{E}'}{c^2} = \frac{E_0^2}{c^2}\left[\left(\gamma\beta\right)^2 - \gamma^2\right] = \frac{E_0^2\left(\beta^2 - 1\right)}{c^2\left(1 - \beta^2\right)} = -\frac{E_0^2}{c^2}$$

20

We can illustrate our two invariants by considering a plane polarized photon which of course will look like a photon in any reference frame. We know that the B field is transverse to the electric field and its B amplitude is E/c. This means our two invariants are both zero for photons in any reference frame. We next consider the field transformations that we worked out previously. Lets consider the case where we have a pure electric field in the z-hat direction in the unprimed frame. The transformed fields into a primed frame moving with a velocity v along the x axis will have a modified E'_z and a new B'_y component. B dot E was zero in the unprimed frame since B was zero and B' dot E' is still zero since our B' field is in the y-hat direction and our E' is still in the y-hat direction. We can also construct out B dot B – E dot E/c^2 invariant in the two frame. The E' field is larger than E by a factor of gamma. But in the unprimed frame B' dot B' is subtracted from the (E' /c)^2 term to give the same value as that in the unprimed frame.

Charged cylinder viewed in moving frame



Transform fields using $\tilde{A}$

$$\vec{E} = \frac{\sigma_0 a}{\varepsilon_0 s}\hat{s} \;\rightarrow V = -\int_a^s \frac{\sigma_0 a}{\varepsilon_0 s''} ds'' = -\frac{\sigma_0 a}{\varepsilon_0}\ln\frac{s}{a}$$

$$\vec{B} = 0 \;\rightarrow \vec{A} = 0$$

$$\begin{pmatrix} V'/c \\ A'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} -\dfrac{\sigma_0 a}{c\varepsilon_0}\ln\dfrac{s}{a} \\ 0 \end{pmatrix} \xrightarrow{s=s'} \begin{matrix} V' = -\dfrac{\gamma\sigma_0 a}{\varepsilon_0}\ln\dfrac{s'}{a} \\ \vec{A}' = \hat{z}\dfrac{\gamma\beta\sigma_0 a}{c\varepsilon_0}\ln\dfrac{s'}{a} \end{matrix}$$

$$\vec{E}' = -\hat{s}\frac{\partial V'}{\partial s'} = \hat{s}\frac{\gamma\sigma_0 a}{\varepsilon_0 s'} \;;\; \vec{B}' = -\hat{\phi}\frac{\partial A'_z}{\partial s'} = -\hat{\phi}\frac{\gamma\beta\sigma_0 a}{c\varepsilon_0 s'}$$

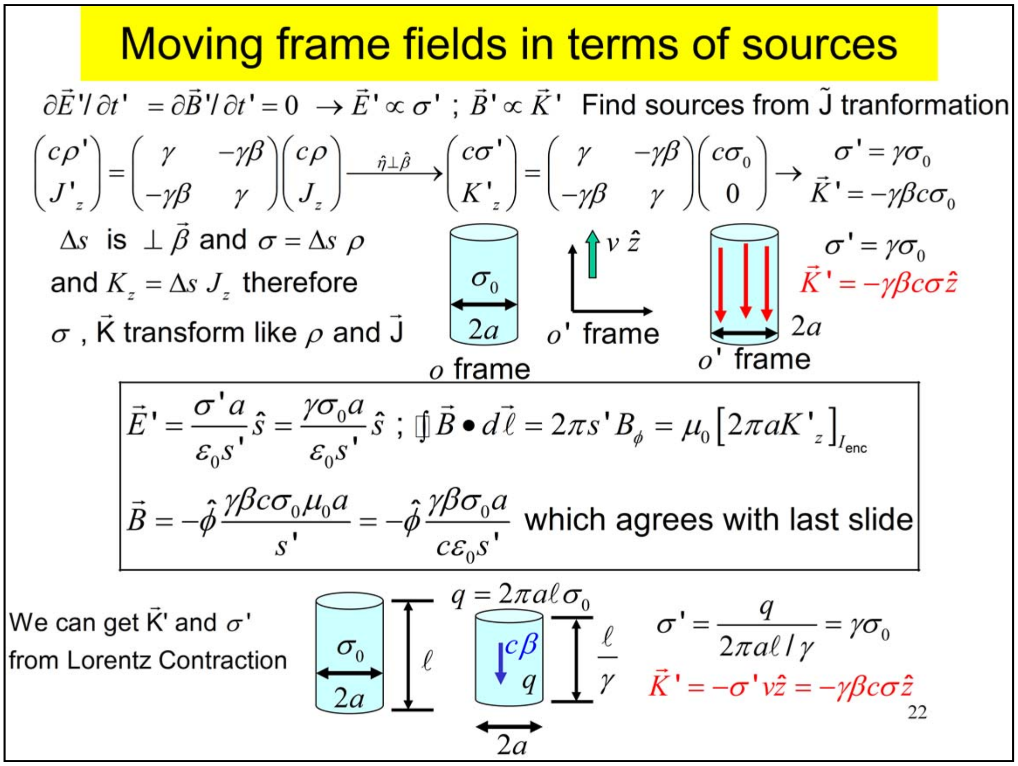
Check invariants

$$\vec{E}\bullet\vec{B} = 0 \;;\; \vec{E}'\bullet\vec{B}' = \hat{s}\frac{\gamma\sigma_0 a}{\varepsilon_0 s'}\bullet\left[-\hat{\phi}\frac{\gamma\beta\sigma_0 a}{c\varepsilon_0}\right] = 0 \;;\; \vec{B}\bullet\vec{B} - \frac{\vec{E}\bullet\vec{E}}{c^2} = -\left(\frac{\sigma_0 a}{c\varepsilon_0 s}\right)^2$$

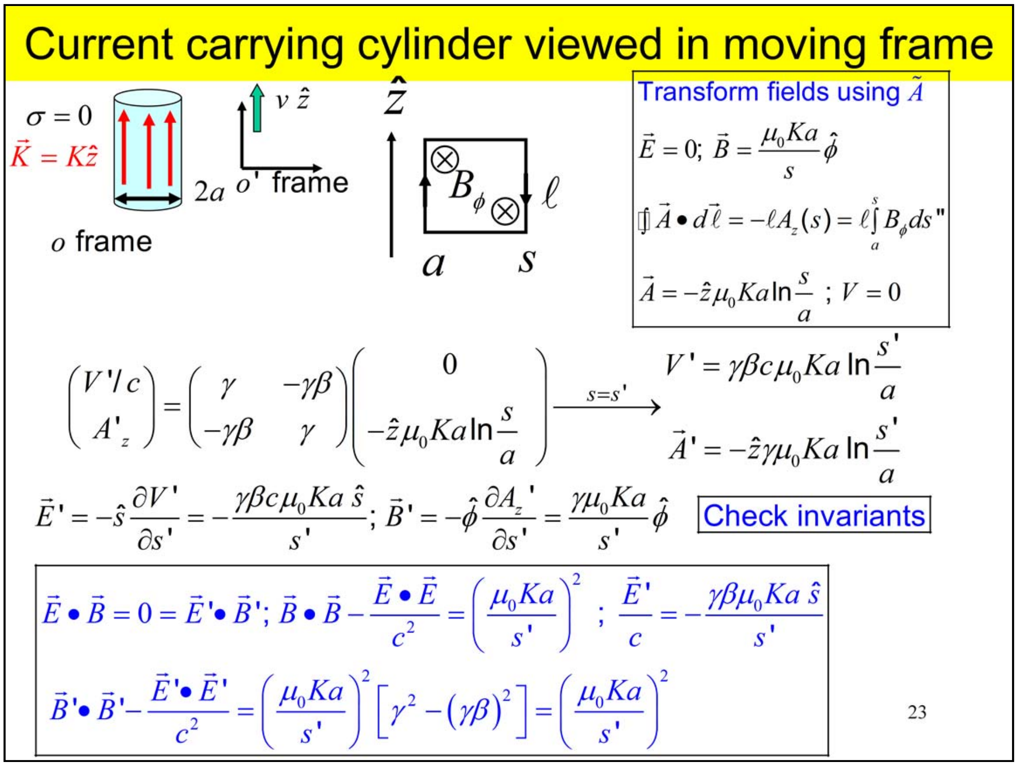
$$\vec{B}'\bullet\vec{B}' - \frac{\vec{E}'\bullet\vec{E}'}{c^2} = \left(\frac{\sigma_0 a}{c\varepsilon_0 s'}\right)^2\left[(\gamma\beta)^2 - \gamma^2\right] = -\left(\frac{\sigma_0 a}{c\varepsilon_0 s'}\right)^2 = -\left(\frac{\sigma_0 a}{c\varepsilon_0 s}\right)^2$$

21

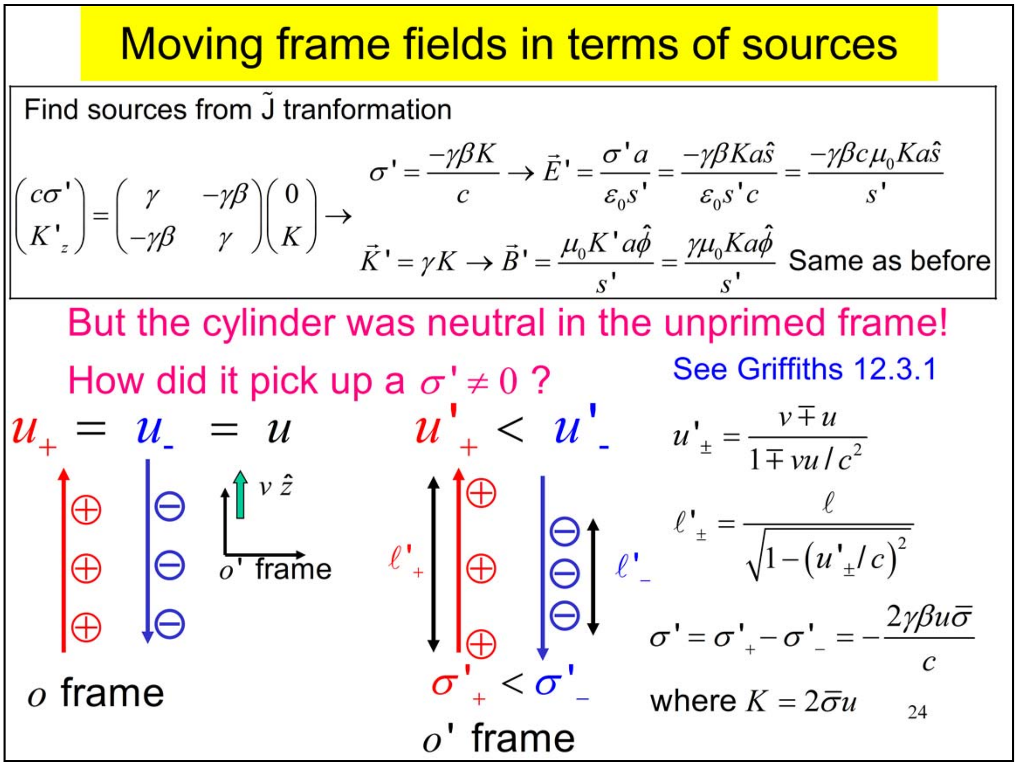
Since Physics 436 is nearly over, I thought it would be good to give two examples of the field invariants where we boost the four potential A-tilde into a primed frame since it involves nearly all aspects of P435 and P436, Our first example is a long cylinder of radius a carrying a surface charge sigma_0. We can use Gauss's law to construct the electric field . We get the scalar potential by - integral E ds" where we zero the potential at s = a. There is no magnetic field and thus vec-A = 0. We could, of course, gauge transform these potentials but this simple form will work. We boost V/c and A_z using a Lorentz boost to a primed frame traveling with velocity v z-hat. A ball rolling along the z axis will be traveling slower in the primed frame so we use negative beta gamma boost. We chose a frame which travels in the z direction to greatly simplify the coordinate transformation. Since x and y are transverse to the boost, they are unaffected and thus s' = s and there is no time dependence in vec-A. We can thus find E' by taking the gradient of V' and the B' from the curl of vec A'. The E'-field is scaled up by a factor of gamma and the B'-field is the same form as a cylinder carrying a surface current. E' and B' are perpendicular so E' dot B' is zero as was E dot B. The B'^2 – (E'/c)^2 is the same as B^2 – (E/c)^2 since the increased E' is compensated with the appearance of B'.



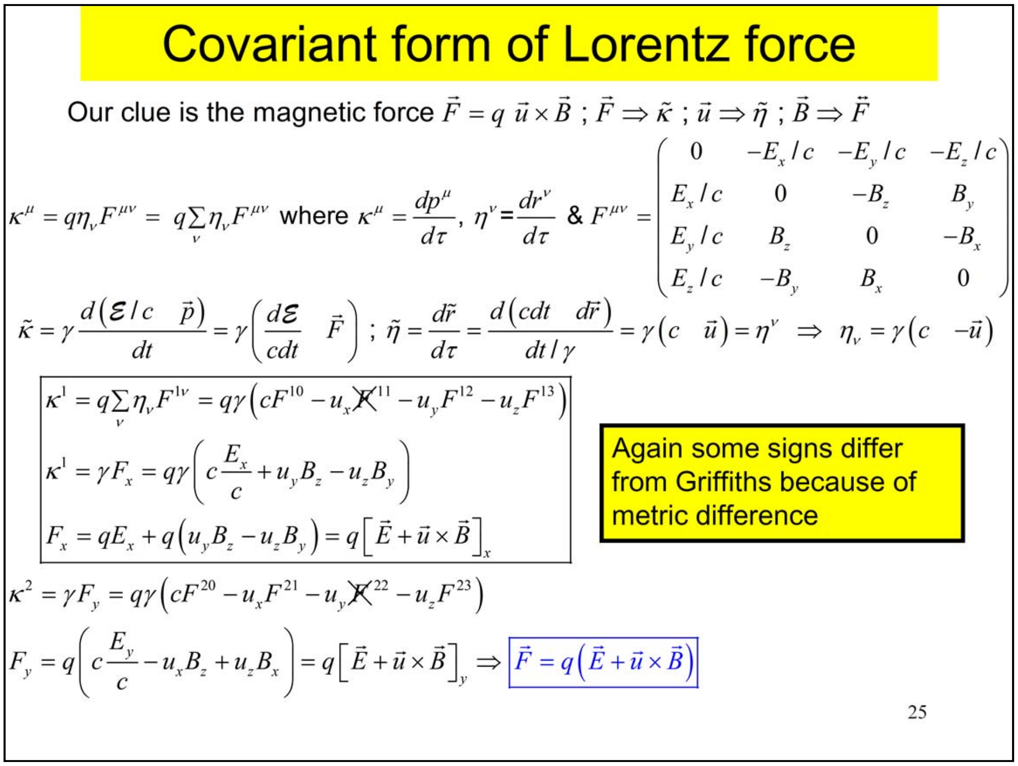
Since all of the relevant coordinates were transverse to the boost direction, the boost adds no time into our potentials and thus we have static fields in the primed frame as well as the unprimed frame. This means there is no Maxwell displacement current contributions for the magnetic field and no Faraday contributions to the electric field. We can thus compute E' and B' from rho' and J' which we can get by boosting rho and J from the unprimed frame. In our example we have a surface charge sigma rather than charge density rho, but we can think of sigma as an infinitesimally thick slab of thickness delta z of rho where sigma = delta z rho. Since delta z is transverse to to the boost direction, there will be no Lorentz contraction and delta z' = delta z. Similarly there is no J but we anticipate a surface current K. We can think of K as an infinitesimally thick J where K = delta z J. So the surface sigma and K transform like the volume rho and J. We thus get a sigma' which is gamma sigma_o and a K' which is gamma sigma v_z. We have already discuss the E-form in terms of sigma, and need to consider the B-field due to K. We simply use Ampere's law which relates the integral of B_phi around a circle to the enclosed current which is just 2 pi a K'. We get exactly the same E' and B' as we did with the four potential. Finally it is very easy to understand our expressions for sigma' and K'. We can think of a length L cylinder as containing a charge of q which is proportional to sigma_0. When viewed in the moving prime frame, the cylinder is Lorentz contracted by a factor of gamma but the charge remains the same. This means that the charge density must increase by a factor of gamma. This enhanced charge density is moving with a velocity of –v z-hat to an observer in the primed frame who therefore sees a surface current of – gamma sigma_0 v z-hat which is exactly what we got from the J-tilde argument for both sigma' and K'.



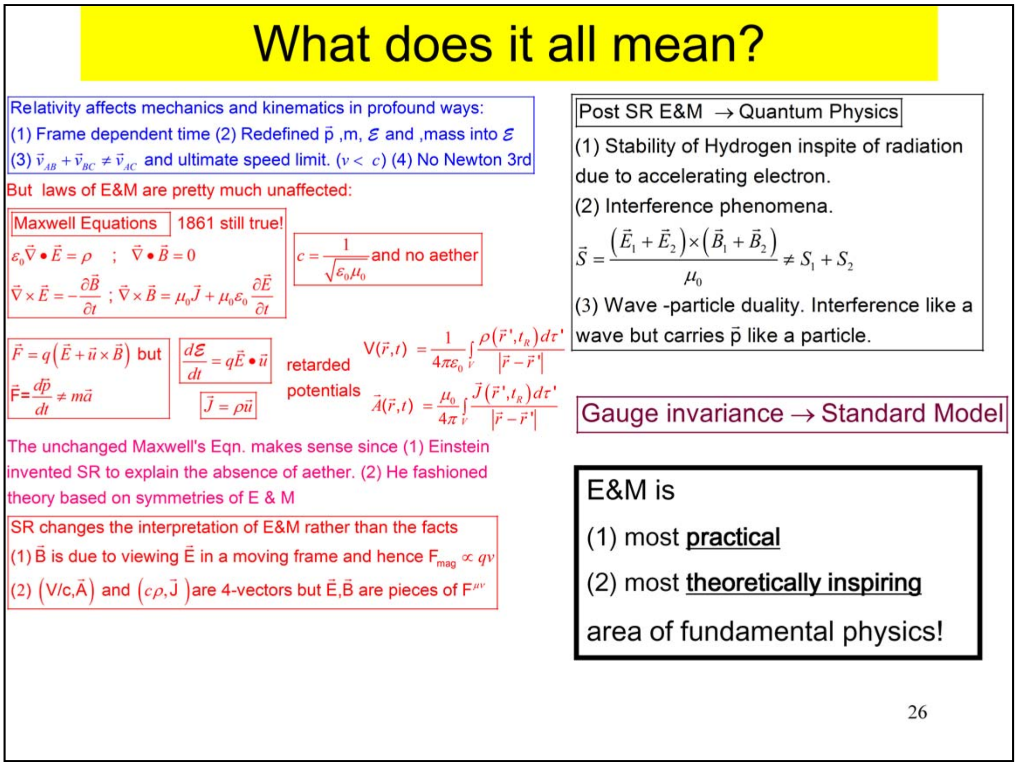
Although our previous example hopefully got most of our points across, it is also interesting to consider the converse case of a neutrally charged cylinder carrying a surface current of K z-hat in the unprimed frame. Again the primed frame is moving with a velocity v along the z axis. We begin by constructing the unprimed A-tilde. Since there is no E-field in the unprimed frame, V = 0. We construct vec-A using the flux method which sets the line integral of A equal to enclosed magnetic flux using Stokes theorem. We do this for a loop which lies transverse to the magnetic field due to the surface current. You can check our vector potential expression by taking its curl. Since our s coordinate is transverse to the boost direction s’ = s and our transformed 4 potential is static meaning the E’ is just the negative gradient of V’ and, as always B’ is just the curl of vec A’. As before the transformed E’ field is transverse to the B’ field and thus the E dot B invariant is automatically satisfied. The B’ field is increased by a factor of gamma compared to B, but the presence of an E’ compensates leaving B^2 – (E/c)^2 invariant. So far so good – No surprises.



The surprise is when build the E' and B' from the transformed sources. This should work since the transformed E' and B' are both static according to the previous slide eliminating any displacement current contributions to B' or Faraday contributions to E'. As before we can boost our sigma and K_z with the same boost as we would use for c rho and J, and we get a sigma' and K' which gives us the same E' and B' fields as we got by boosting the four potential. Again so far so good. But how did our neutral cylinder in the unprimed frame pick up a charge density in the primed frame?? There is a nice discussion of this phenomena in Section Griffiths 12.3.1 which uses a simple model for the surface current in the unprimed frame. In this model, we think of the negative charges moving with an equal and opposite velocity as the positive charges. Of course in a real wire, the negative charges (electrons) move while the positive charge (protons bound into ions) do not but this elegant model is much easier to calculate. In my cartoon, this velocity is given the symbol u and we think of the sigma as due to three evenly space negative charges and three evenly spaced positive charges. We next consider what this model looks like in the primed frame. The positive charges are moving slower in the primed frame since the primed frame is traveling in the direction of the positive. Similarly the negative charges are moving faster in the primed frame. The spacing between our three negative charges is Lorentz contracted more than the spacing between our three positive charges meaning sigma'_- is larger than sigma'_+ meaning a net negative charge density which creates an electrical field which points into the cylinder. This all might seem pretty implausible since the electron speed u in the conductor is very small but our induced sigma' is also down by two powers of c as well. Griffiths 12.3.1 uses the relative velocity and Lorentz contraction factor expressions to recover our sigma' expression in this model. Griffiths uses this model, to argue that the current in a wire when viewed in a moving frame creates a net charge density to the wire which attracts a test charge. The Lorentz force is essentially the force between the test charge and the net wire charge density. Hence magnetism is really just electrostatics viewed through relativity.



We have showed that Maxwell's equations are all relativistically-ready but what about the force laws of electrodynamics summarized by the Lorentz force? On this slide we show how to write the Lorentz force in covariant (i.e. 4-vector) form. We use the usual (classical) form as inspiration where the force (or rate of change of momentum) is the charge times velocity cross field. We have discussed how the velocity can be converted to a 4-velocity which is the rate of change of 4-interval with respect to proper time. The field can be replaced by the field tensor F^(mu nu) which leaves us with a 4-force kappa which (in the same spirit) is the change of the 4 momentum with respect to proper time. This is a fairly obvious 4-vector version of the Lorentz force but does it give the classical Lorentz force or some relativistic correction to the Lorentz force? Our first step is to write the 4-force, 4-velocity and field tensor in terms of classical forces, classical velocities, and classical electromagnetic fields. Note we use the covariant rather than contravariant components of the 4-velocity which brings in an extra (-) sign in the velocity piece. We next check a few components of the 4-force starting the x-component. We write out the 4 terms involved with the nu summation in terms of classical quantities. We can convert this to ordinary force by dividing by gamma and viola we get the x-component of the classical Lorentz force written in terms of 3-vector cross products. We also check the y-component which again agrees with the classical Lorentz force. Many of our signs differ from the text because of Griffiths' unfortunate (IMHO) metric choice. So again the traditional electromagnetic forces are already relativistic-ready.



So what did we learn in this chapter on relativistic electrodynamics? The main thing we learned is that the chapter title is misleading – there is no "relativistic" electrodynamics. Maxwell's equations and the Lorentz force are correct as originally stated by Maxwell, Faraday, Ampere etc. The real changes due to relativity are changes to classical mechanics not classical electrodynamics. Special relativity has basically clarified the interpretation of electrodynamics rather than changing the "facts" of electrodynamics. In fact, historically Maxwell electrodynamics served as a template for relativity. As you learn more physics you will see electrodynamics is very much a template for quantum physics, field theory, and modern gauge theories of subatomic physics as well. Hence electrodynamics is not only the most practical physics but the most theoretically inspiring.