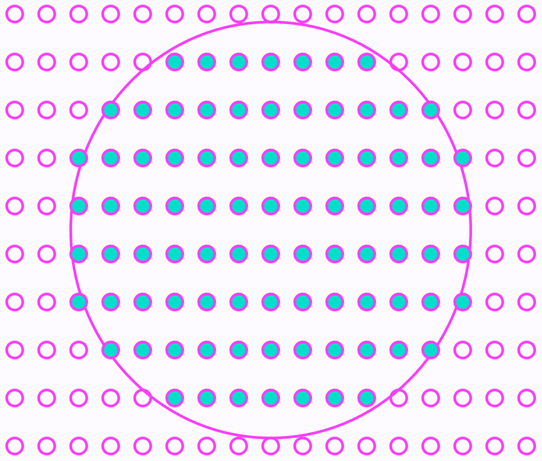
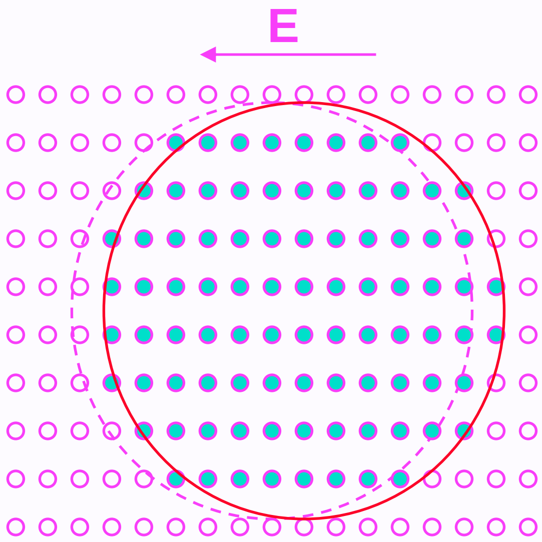
Part II - Electronic Properties of Solids

Lecture 13: The Electron Gas Continued

(Kittel Ch. 6)



Equilibrium - no field



With applied field

Outline

- From last time:
    - Success of quantum mechanics
    - Pauli Exclusion Principle, Fermi Statistics
    - Energy levels in 1 and 3 dimensions

    - Density of States, Heat Capacity

- Today:
    - Fermi surface
    - Transport
        - Electrical conductivity and Ohm's law
        - Impurity, phonon scattering
        - Hall Effect
        - Thermal conductivity
        - Metallic Binding
- (Read Kittel Ch 6)

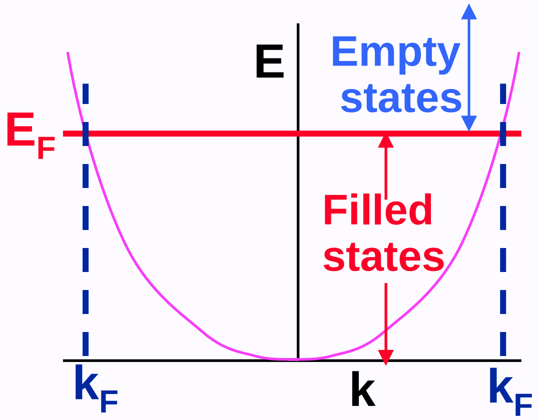
Electron Gas in 3 dimensions

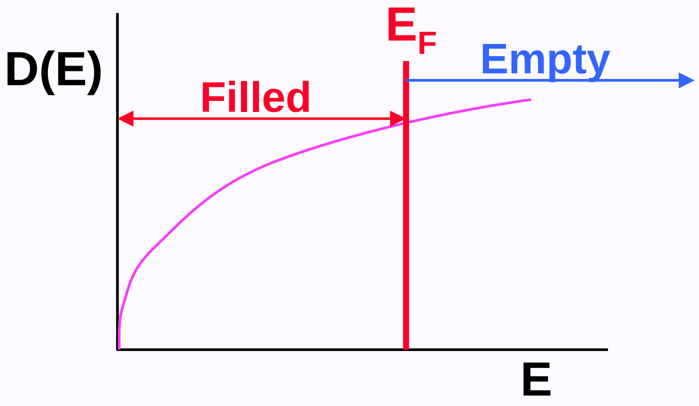
- Recall from last lecture:
- Energy vs k

$$E\ (k) = ((\hbar^2/2m)\,(k_x^2 + k_y^2 + k_z^2\ ) = (\hbar^2/2m)k^2$$

- Density of states

$$D(E) = (1/2\pi^2)\ E^{1/2}(\hbar^2/2m)^{-3/2} \sim E^{1/2}$$



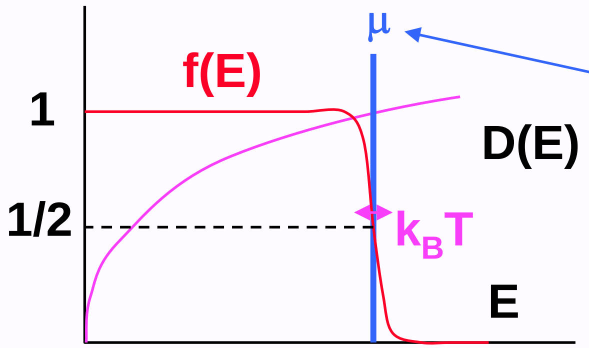


•Electrons obey exclusion Principle:
The lowest energy possible is for all states filled up to the Fermi momentum k_F and Fermi energy $E_F = (\hbar^2/2m)k_F^2$ given by
$k_F = (3\pi^2\, N_{elec}/V\ )^{1/3}$ and $E_F = (\hbar^2/2m)\,(3\pi^2\, N_{elec}/V\ )^{2/3}$

Fermi Distribution

- At finite temperature, electrons are not all in the lowest energy states. Thermal energy causes states to be partially occupied.
- Fermi Distribution (Kittel appendix)

$$f(E) = 1/[\exp((E-\mu)/k_BT) + 1]$$



Chemical potential for electrons = Fermi energy at T=0

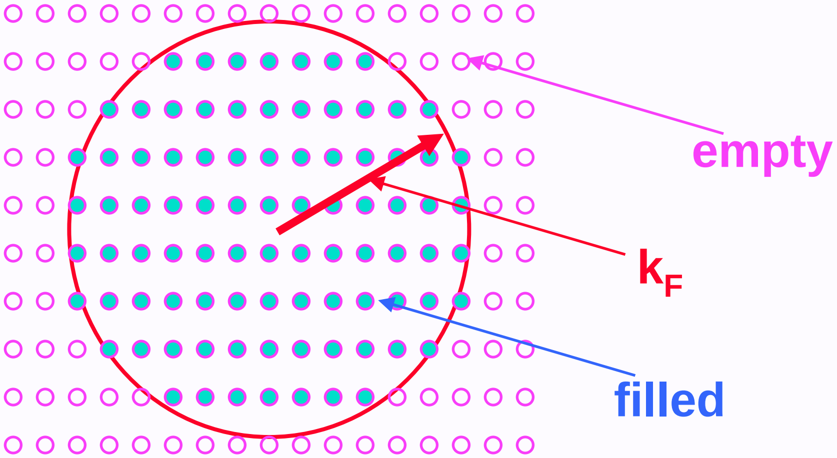
- For typical metals the Fermi energy is much greater than ordinary temperatures. Example:

For Al, E_F = 11.6 eV, i.e., $T_F = E_F/k_B = 13.5 \times 10^4$ K

- At ordinary temperature, the only change in the occupation of the states is very near the chemical potential μ. States are filled for states with $E << \mu$, and empty for states with $E >> \mu$.
- Heat capacity $C = dU/dT \sim N_{elec} k_B (T/ T_F)$

Electrical Conductivity & Ohm’s Law

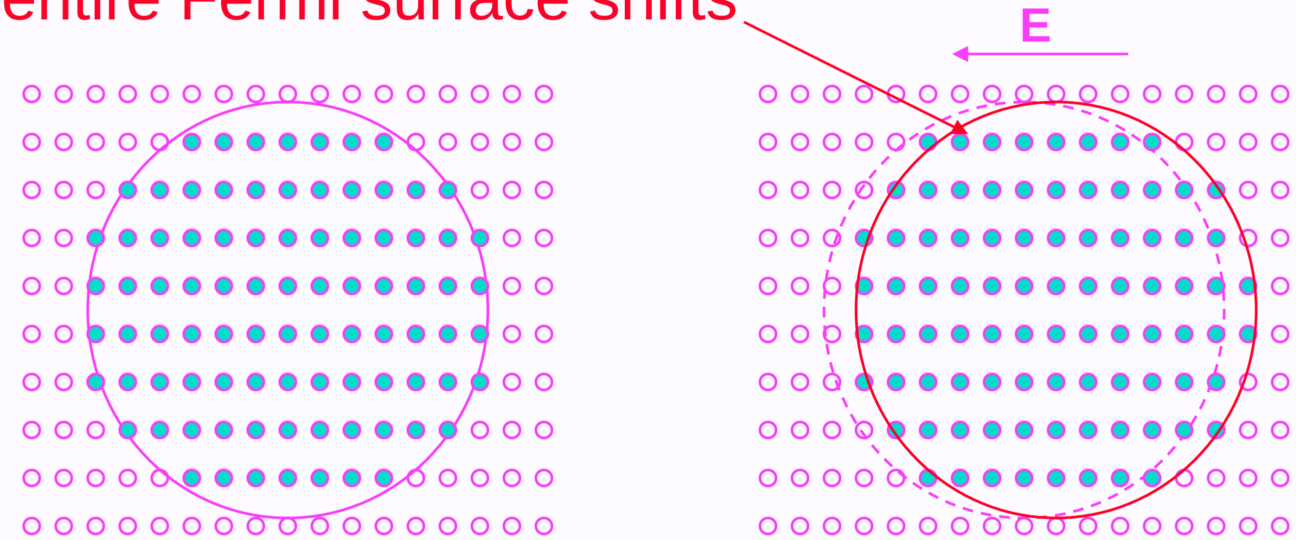
- The filling of the states is described by the Fermi surface – the surface in k-space that separates filled from empty states
- For the electron gas this is a sphere of radius k_F.



Lowest energy state filled for states with $k < k_F$, i.e., $E < E_F$

Electrical Conductivity & Ohm's Law

- Consider electrons in an external field E. They experience a force F = -eE
- Now $F = dp/dt = \hbar\, dk/dt$, since $p = \hbar k$
- Thus in the presence of an electric field all the electrons accelerate and the k points shift, i.e., the entire Fermi surface shifts

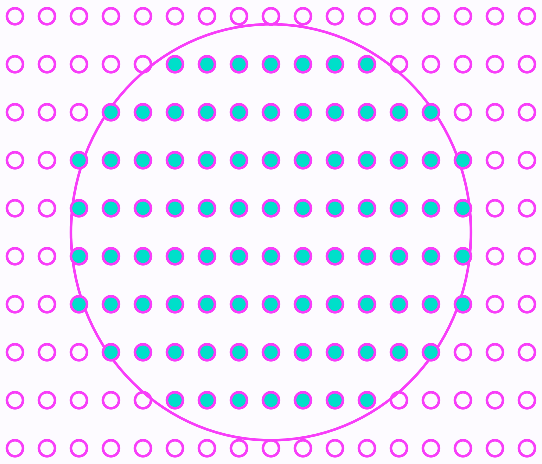


Equilibrium - no field

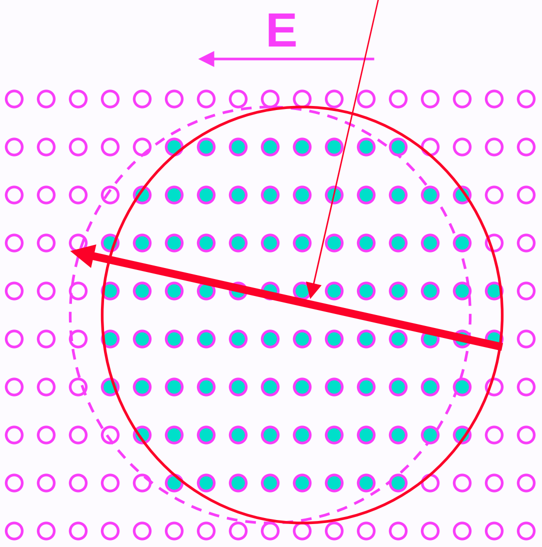
With applied field

Electrical Conductivity & Ohm’s Law

- What limits the acceleration of the electrons?
- Scattering increases as the electrons deviate more from equilibrium
- After field is applied a new equilibrium results as a balance of acceleration by field and scattering



Equilibrium - no field



With applied field

Electrical Conductivity and Resitivity

- The conductivity σ is defined by $j = \sigma E$,
  where j = current density
- How to find σ?
- From before $F = dp/dt = m\, dv/dt = \hbar\, dk/dt$
- Equilibrium is established when the rate that k increases due to E equals the rate of decrease due to scattering, then $dk/dt = 0$
- If we define a scattering time τ and scattering rate$1/\tau$
  $$\hbar\,(dk/dt + k/\tau) = F = q E \quad (q = \text{charge})$$
- Now $j = n q v$ (where n = density) so that
  $$j = n q (\hbar k/m) = (n q^2/m)\, \tau E$$
  $$\Rightarrow \boxed{\sigma = (n q^2/m)\, \tau}$$
- Resistance: $\rho = 1/\sigma \propto m/(n q^2 \tau)$

Note: sign of charge does not matter

Scattering mechanisms

- Impurities - wrong atoms, missing atoms, extra atoms, ….

  Proportional to concentration

- Lattice vibrations - atoms out of their ideal places

  Proportional to mean square displacement

- This also applies to a crystal (not just the electron gas) using the fact that there is no scattering in a perfect crystal as discussed in the next lectures

Electrical Resitivity

- Resistivity ρ is due to scattering: Scattering rate inversely proportional to scattering time τ

$$\rho \propto \text{scattering rate} \propto 1/\tau$$

- Matthiesson's rule - scattering rates add

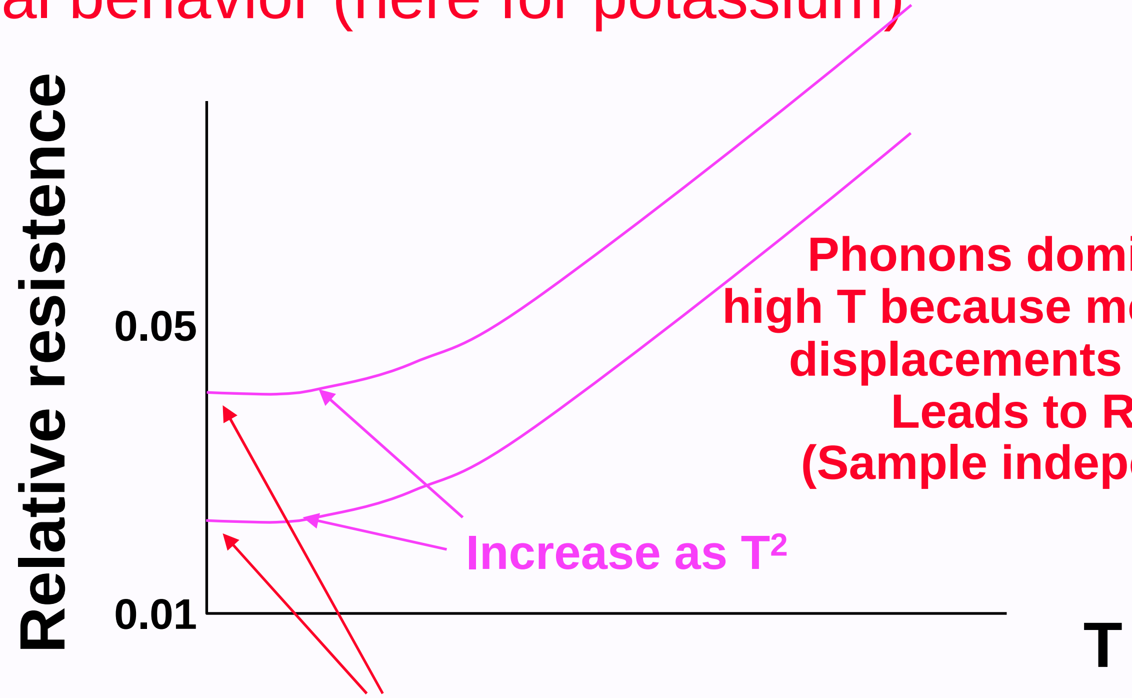
$$\rho = \rho_{\text{vibration}} + \rho_{\text{impurity}} \propto 1/\tau_{\text{vibration}} + 1/\tau_{\text{impurity}}$$

Temperature dependent
$\propto \langle u^2 \rangle$

Temperature independent
- sample dependent

Electrical Resitivity

- Consider relative resistance R(T)/R(T=300K)
- Typical behavior (here for potassium)



Phonons dominate at
high T because mean square
displacements $\langle u^2 \rangle \propto T$
Leads to $R \propto T$
(Sample independent)

Increase as T^2

Inpurity scattering dominates
at low T in a metal
(Sample dependent)

Interpretation of Ohm's law
Electrons act like a gas

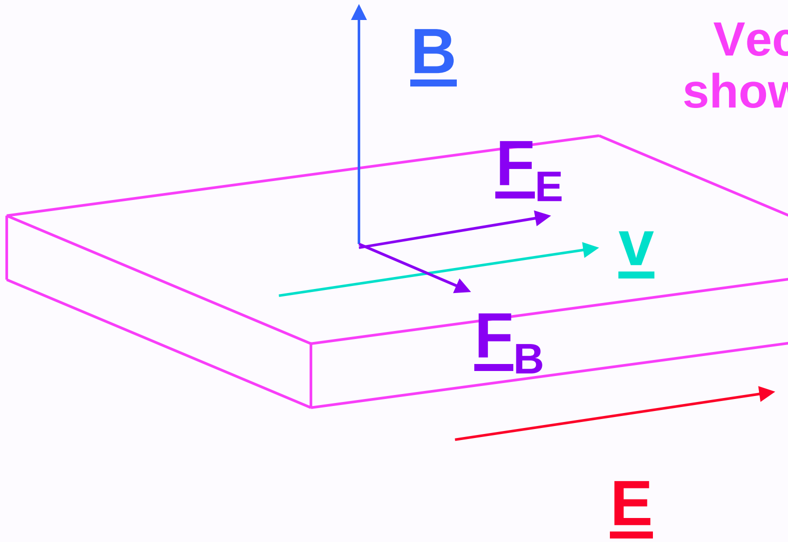
- A electron is a particle - like a molecule.
- Electrons come to equilibrium by scattering like molecules (electron scattering is due to defects, phonons, and electron-electron scattering).
- Electrical conductivity occurs because the electrons are charged, and it shows the electrons move and equilibrate
- What is different from usual molecules?
  Electrons obey the exclusion principle. This limits the allowed scattering which means that electrons act like a weakly interacting gas.

Hall Effect I

- Electrons moving in an electric and a perpendicular magnetic field
- Now we must carefully specify the vector force

$\underline{\mathbf{F}} = q(\underline{\mathbf{E}} + (1/c)\, \underline{\mathbf{v}} \times \underline{\mathbf{B}})$ (note: c → 1 for SI units)

(q = -e for electrons)

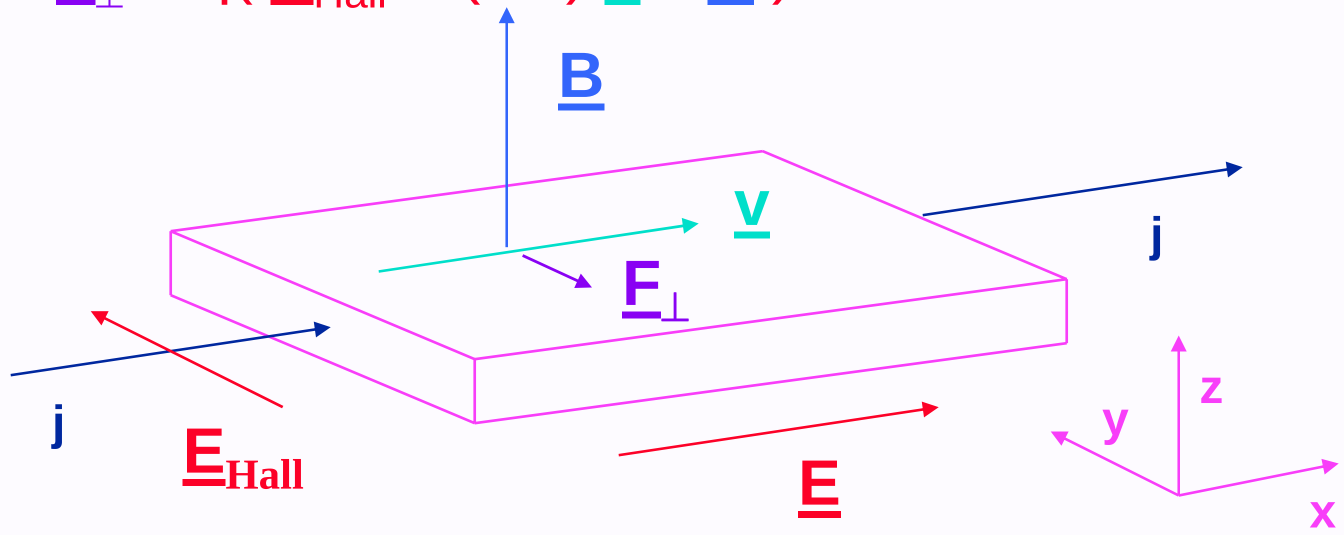


Vector directions shown for positive q

Hall Effect II

- Relevant situation: current $j = \sigma E = nqv$ flowing along a long sample due to the field **E**
- But NO current flowing in the perpendicular direction
- This means there must be a Hall field $\mathbf{E}_{\text{Hall}}$ in the perpendicular direction so the net force $\mathbf{F}_{\perp} = 0$

$$\mathbf{F}_{\perp} = q(\mathbf{E}_{\text{Hall}} + (1/c)\, \mathbf{v} \times \mathbf{B}) = 0$$



Hall Effect III

- Since

$$\underline{F}_{\perp} = q(\underline{E}_{Hall} + (1/c)\ \underline{v} \times \underline{B}) = 0 \quad \text{and } v = j/nq$$

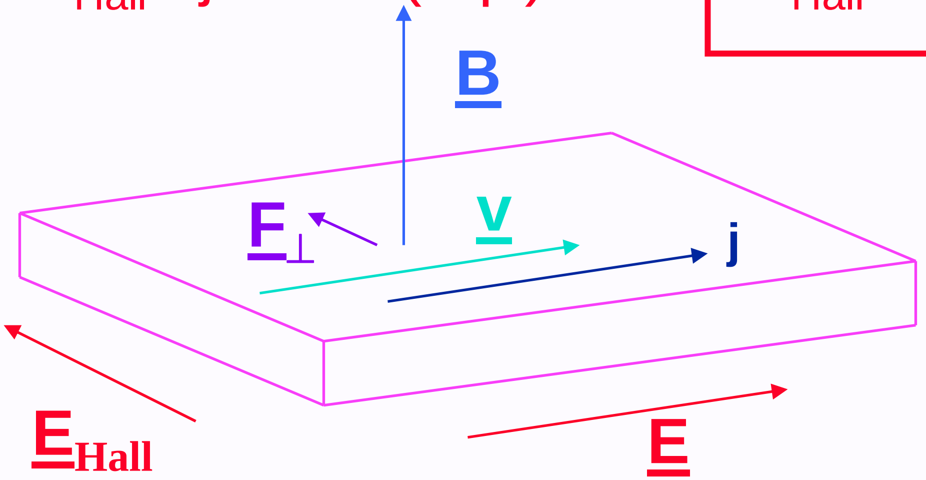
then defining $v = (\underline{v})_x$, $E_{Hall} = (\underline{E}_{Hall})_y$, $B = (\underline{B})_z$,

$$E_{Hall} = - (1/c)\ (j/nq)\ (- B)$$

Sign from cross product

and the Hall coefficient is

$$R_{Hall} = E_{Hall} / j B = 1/(nqc) \quad \text{or} \quad \boxed{R_{Hall} = 1/(nq) \ \text{ in SI}}$$



Hall Effect IV

- Finally, define the Hall resistance as

$$\rho_{Hall} = R_{Hall}\, B = E_{Hall} / j$$

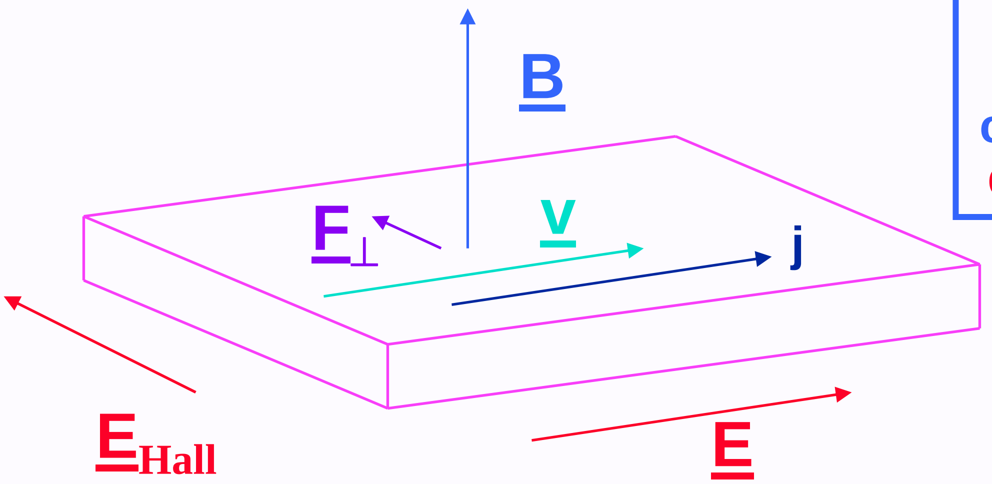
Each of these quantities can be measured directly

which has the same units as ordinary resistivity

- $R_{Hall} = E_{Hall} / j\, B = 1/(nq)$

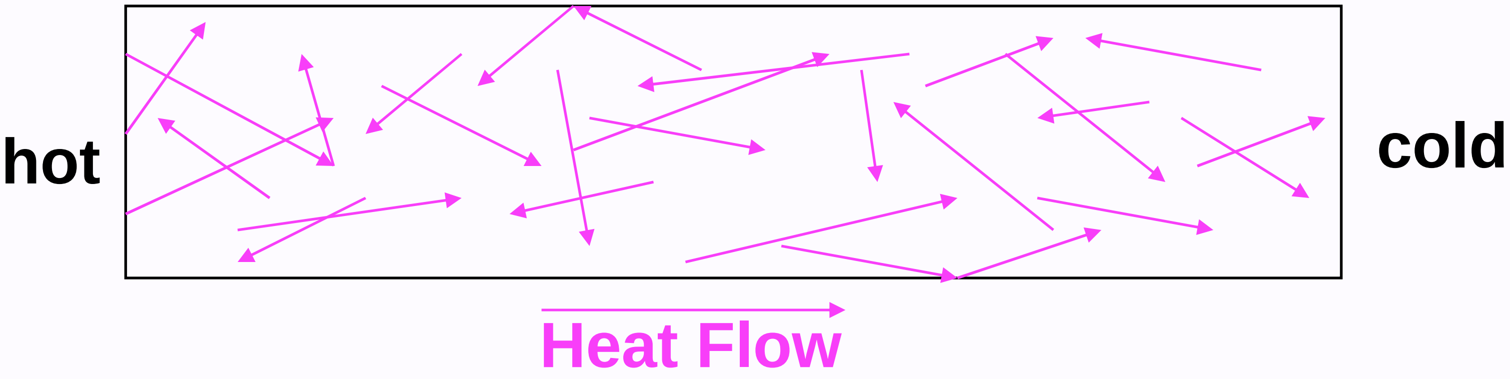
Note: R_{Hall} determines sign of charge q

Since magnitude of charge is known R_{Hall} determines density n



Heat Transport due to Electrons

- A electron is a particle that carries energy - just like a molecule.
- Electrical conductivity shows the electrons move, scatter, and equilibrate
- What is different from usual molecules? Electrons obey the exclusion principle. This limits scattering and helps them act like weakly interacting gas.



Heat Transport due to Electrons

- Definition (just as for phonons):
  $j_{thermal}$ = heat flow (energy per unit area per unit time)
  $= - K\, dT/dx$
- If an electron moves from a region with local temperature T to one with local temperature $T - \Delta T$, it supplies excess energy $c\, \Delta T$, where c = heat capacity per electron. (Note ΔT can be positive or negative).
- On the average for a thermal :
  $\Delta T = (dT/dx)\, v_x\, \tau$, where τ = mean time between collisions
- Then $j = - \underbrace{n\ \ v_x}_{\text{Flux}}\, c\, v_x\, \tau\, dT/dx = - n\ \ c\, v_x^2\, \tau\, dT/dx$

Density

Flux

Electron Heat Transport - continued

- Just as for phonons:
  Averaging over directions gives $(v_x^2)_{\text{average}} = (1/3) v^2$
  and

$$j = - (1/3)\, n\, c\, v^2\, \tau\, dT/dx$$

- Finally we can define the mean free path $L = v\,\tau$
  and $C = nc$ = total heat capacity,
  Then

$$j = - (1/3)\, C\, v\, L\, dT/dx$$

and

$K = (1/3)\, C\, v\, L = (1/3)\, C\, v^2\, \tau$ = thermal conductivity

(just like an ordinary gas!)

Electron Heat Transport - continued

- What is the appropriate v?
- The velocity at the Fermi surface = v_F
- What is the appropriate τ ?
- Same as for conductivity (almost).

- Results using our previous expressions for C:

$$K = (\pi^2/3)\ \ (n/m)\ \tau\ k_B^2\ T$$

- Relation of K and σ -- From our expressions:

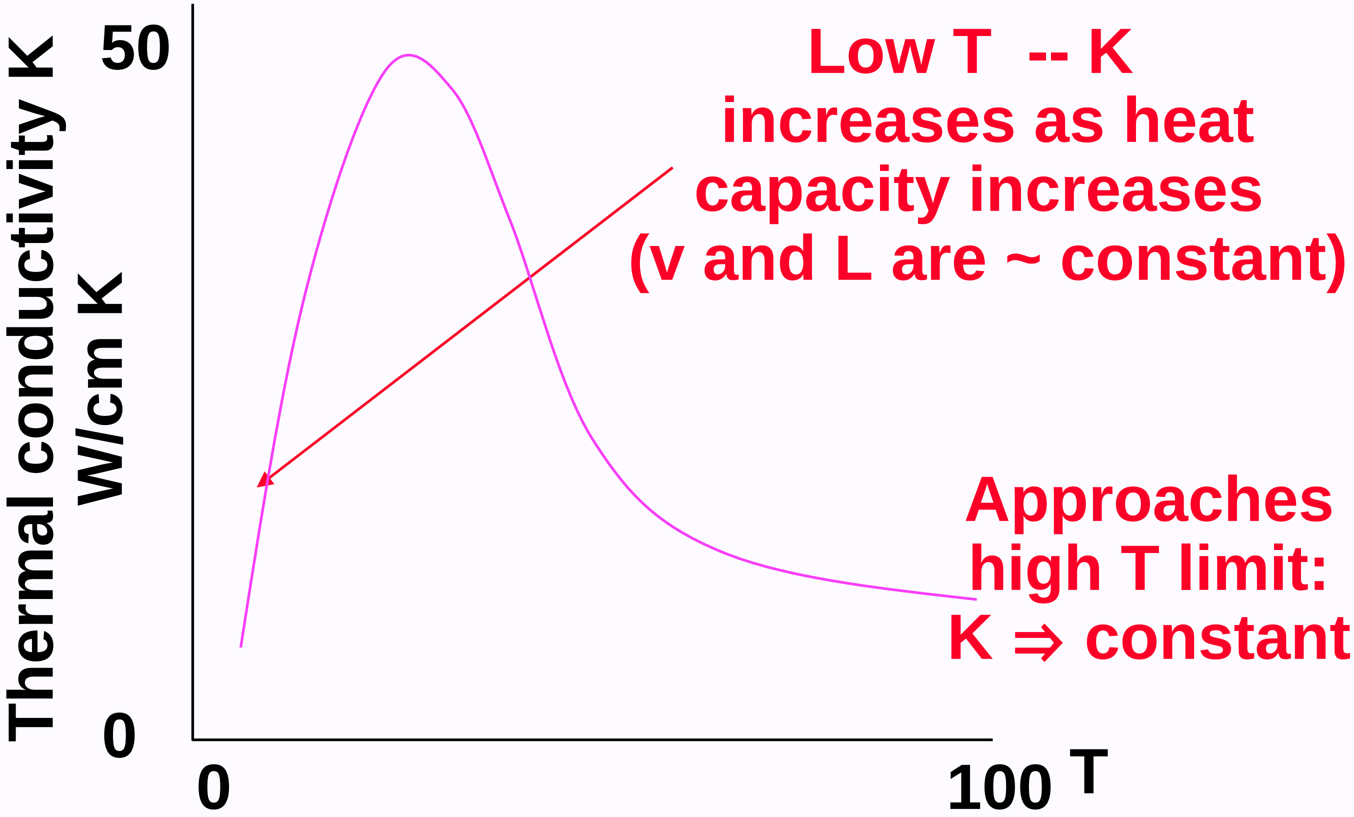
$$K / \sigma\ = (\pi^2/3)\ (k_B/e)^2\ T$$

- This justifies the Weidemann-Franz Law that

$$K / \sigma\ \propto T$$

Electron Heat Transport - continued

- K $\propto$ σ T
- Recall $\sigma \to$ constant as T $\to$ 0, $\sigma \to$ 1/T as T $\to$ large



Low T -- K increases as heat capacity increases (v and L are ~ constant)

Approaches high T limit: K $\Rightarrow$ constant

Electron Heat Transport - continued

- Comparison to Phonons

Electrons dominate in good metal crystals

Comparable in poor metals like alloys

Phonons dominate in non-metals

Metallic Binding

- (Treated only in problems in Kittel)
- Electron gas kinetic energy is positive, i.e., replusive.

  See homework for E, pressure, bulk modulus

  Key point: $E_{kinetic} \propto (1/V)^{2/3}$
- What is the attraction that holds metals together?

  Coulomb attraction for the nuclei

  NOT included in gas so far - must be added
- Energy of point nuclei in uniform electron gas:

  Key point: $E_{Coulomb} \propto -(1/V)^{1/3}$

  Approximate expressions in Kittel problem 8

  Energy per electron:

  $E_{Coulomb} \propto -1.80/r_s$ Ryd, where $(4\pi/3)r_s{}^3 = V$
- Net effect is metallic binding

Where can the electron gas be found?

- In semiconductors!
  More later - in doped semiconductors, the extra electrons (or missing electrons) can act like an electron gas in a background
- Where can 1d or 2d gas be found?
  In semiconductor structures!
  Layers of GaAs and AlAS can make nearly Ideal 2d gasses
  1d “wires” can also be made
- More later

Summary

- **Electrical Conductivity - Ohm's Law**

  $\sigma = (n\, q^2/m)\, \tau \qquad \rho = 1/\sigma$

- **Hall Effect**

  $\rho_{Hall} = R_{Hall}\, B = E_{Hall} / j$

  ρ and ρ_{Hall} determine n and the charge of the carriers

- **Thermal Conductivity**

  $K = (\pi^2/3)\ (n/m)\, \tau\, k_B^2\, T$

  Weidemann-Franz Law:

  $K / \sigma = (\pi^2/3)\, (k_B/e)^2\, T$

- **Metallic Binding**

  Kinetic repulsion

  Coulomb attraction to nuclei

  (not included in gas model - must be added)

Next time

- EXAM   Wednesday, October 11
- Next week: Electrons in crystals
  - Energy Bands
  - We will use many ideas from the understanding of crystals and lattice vibrations to describe electron waves in a periodic crystal!
  - (Read Kittel Ch 7)

Comments on Exam

- Three types of problems:

- Short answer questions
- Order of Magnitudes
- Essay question
- Quantitative problems