1. Free electron energies on the square lattice in the empty lattice approximation
   (a) Show for a simple square lattice (two dimensions) that the kinetic energy of a free electron at a corner of the (first) Brillouin zone is higher than that of an electron at the midpoint of a side face of the Brillouin zone by a factor of 2.
   (b) What is the corresponding factor for a simple cubic lattice (i.e., in three dimensions)?

2. The primitive translation vectors of the 3D hexagonal/honeycomb lattice may be taken as
   \[ \mathbf{a}_1 = (\sqrt{3}a/2)\hat{x} + (a/2)\hat{y}; \quad \mathbf{a}_2 = -(\sqrt{3}a/2)\hat{x} + (a/2)\hat{y}; \quad \mathbf{a}_3 = c\hat{z}. \] (1)
   (a) Show that the volume of the primitive cell is \((3^{1/2}/2)a^2c\).
   (b) Show that the primitive basis vectors of the reciprocal lattice are
   \[ \mathbf{b}_1 = (2\pi/\sqrt{3}a)\hat{x} + (2\pi/a)\hat{y}; \quad \mathbf{b}_2 = -(2\pi/\sqrt{3}a)\hat{x} + (2\pi/a)\hat{y}; \quad \mathbf{b}_3 = (2\pi/c)\hat{z}. \] (2)
   Show that this implies that the lattice is its own reciprocal, except for a rotation and re-scaling of axes.
   (c) Describe and sketch the (first) Brillouin zone of the hexagonal space lattice.

3. For the primitive lattice in the previous problem draw (at least) a 5 × 5 set of unit cells/lattice points in the \(\vec{a}_1\) and \(\vec{a}_2\) plane, i.e., draw lattice points that are constructed as \(\vec{R} = n\vec{a}_1 + m\vec{a}_2\). Draw lines representing the 2d Miller indices (10), (01), (11), (21), (12), (41).

4. Show that the generically in 3d the volume of the (first) Brillouin zone is \((2\pi)^3/V_c\), where \(V_c\) is the volume of a real-space crystal primitive cell. Hint: The volume of a Brillouin zone is equal to the volume of the primitive parallelepiped in reciprocal space. Recall the vector identity \((\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))\mathbf{a}\).

5. Consider a periodic potential \(V(x, y) = \sin(2\pi x/a) + \sin(4\pi y/a)\).
   (a) Write down a set of primitive lattice vectors \(\vec{a}_1, \vec{a}_2\) that define the period of this potential.
   (b) Fourier expand \(V(x, y) = \sum_{\vec{G}} V_{\vec{G}} e^{i\vec{G} \cdot \vec{r}}\) by determining the set \(\vec{G}\) and the coefficients \(V_{\vec{G}}\).
   (c) Show that the \(\vec{R} \cdot \vec{G} = 2\pi q\) for an integer \(q\) where \(\vec{R} = n\vec{a}_1 + m\vec{a}_2\).
   (d) Repeat steps (a) through (c) for the potential \(V(x, y) = \sin(2\pi(x+y)/a) + \cos(2\pi(x-2y)/a)\).
   (e) For both sets of potentials convert the continuous periodic potential into a discrete set of lattice points (draw a subset of these points which illustrates the lattice structure). This can be done by choosing a point somewhere inside a cell of each periodic potential and translating that point by the respective \(\vec{a}_1, \vec{a}_2\). How does the resulting lattice change if we pick a different starting point within a cell of the periodic potential?

6. Using the \(\vec{G}\) derived in the previous problem
   (a) Write down an explicit form for the Bloch functions for both potentials and show that they are periodic when translated by a lattice translations \(\vec{R} = n\vec{a}_1 + m\vec{a}_2\) for all integers \(n, m\).
   (b) Multiply each Bloch function by a suitable plane wave so that it represents an eigenstate of \(H = \frac{p^2}{2m} + V(x, y)\).
   (c) Determine the allowed range for the wave vector \(\vec{q}\) entering the plane wave piece and draw the 2d Brillouin zone (which contains all unique values of \(q\)) for each potential.
   (d) Find translation operators \(T(\vec{d}) = e^{i\vec{d} \cdot \vec{p}/\hbar}\) that commute with the Hamiltonian, i.e. determine the smallest allowed values of the vector \(\vec{d}\) such that \(T(\vec{d})\) commutes with \(H = \frac{p^2}{2m} + V(x, y)\) for the two choices of \(V(x, y)\). Show explicitly that \([T(\vec{d}), H] = 0\). Show that the eigenstates determined in part (c) are eigenstates of these translation operators.