Random Number Generation (RNG)

read "Numerical Recipes" on random numbers and the chi-squared test.

Today we discuss how to generate and test random numbers.

What is a random number?

- A single number is not random. Only an infinite sequence can be described as random.
- Random means the absence of order. (Negative property).
- Can an intelligent gambler make money by betting on the next numbers that will turn up?
- All subsequences are equally distributed. This is the property that MC uses to do integrals and that you will test for in homework.

"Anyone who considers arithmetical methods of random digits is, of course, in a state of sin." *John von Neumann (1951)*

"Random numbers should not be chosen at random." Knuth (1986)

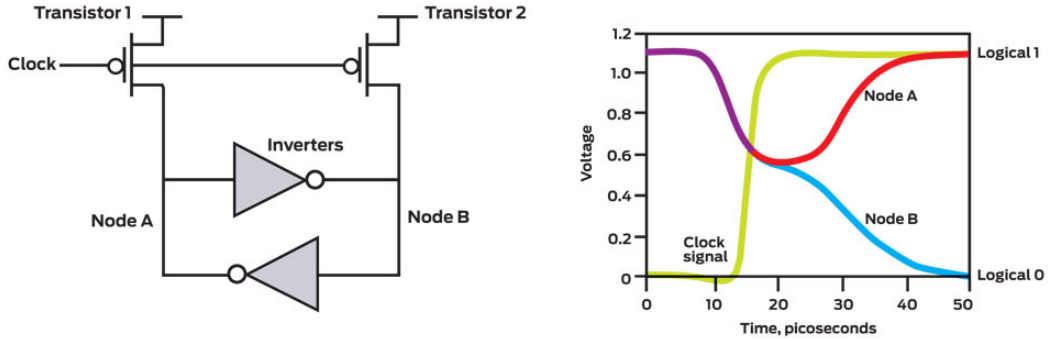
Random numbers on a computer

- Truly Random - the result of a physical process such as timing clocks, circuit noise, Geiger counts, bad memory
  - Too slow (we need 10^{10}/sec) and expensive
  - Low quality
  - Not reproducible. But new INTEL chip does this for security
- Pseudo-random. *prng* (pseudo means *fake*)
  - Deterministic sequence with a repeat period but with the appearance of randomness (if you don't know the algorithm).
- Quasi-random (quasi means *almost random*)
  - "half way" between random and a uniform grid.
  - Meant to fill space with max. distance between space to fill "holes" sequentially (e.g., [0,100] as 0,1,2,3,…,100).

If doing a low-D integral, QRNG gives up correlation so the sequence will be more uniform than PRNG and converge to mean value quicker than $N^{-1.2}$.

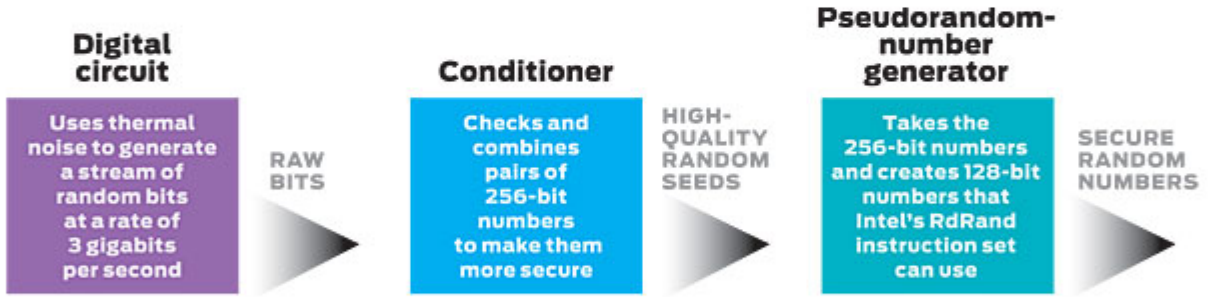
Intel's RNG: Ivy Bridge

- Article describing it: 3 Gbits/sec
- http://spectrum.ieee.org/computing/hardware/behind-intels-new-randomnumber-generator/0



UNCERTAIN CIRCUITS: When transistor 1 and transistor 2 are switched on, a coupled pair of inverters force Node A and Node B into the same state [left]. When the clock pulse rises [yellow, right], these transistors are turned off. Initially the output of both inverters falls into an indeterminate state, but random thermal noise within the inverters soon jostles one node into the logical 1 state and the other goes to logical 0. © 2011 IEEE Spectrum magazine

CLOSE ✕



THREE-LAYER NUMBERS: Intel's Bull Mountain random-number generator prevents bias and correlation with a three-step process. First, a digital circuit generates a stream of raw random bits. Next, a conditioner generates healthy random seeds. Third, a pseudorandom-number generator readies the numbers for use in software. © 2011 IEEE Spectrum magazine

CLOSE ✕

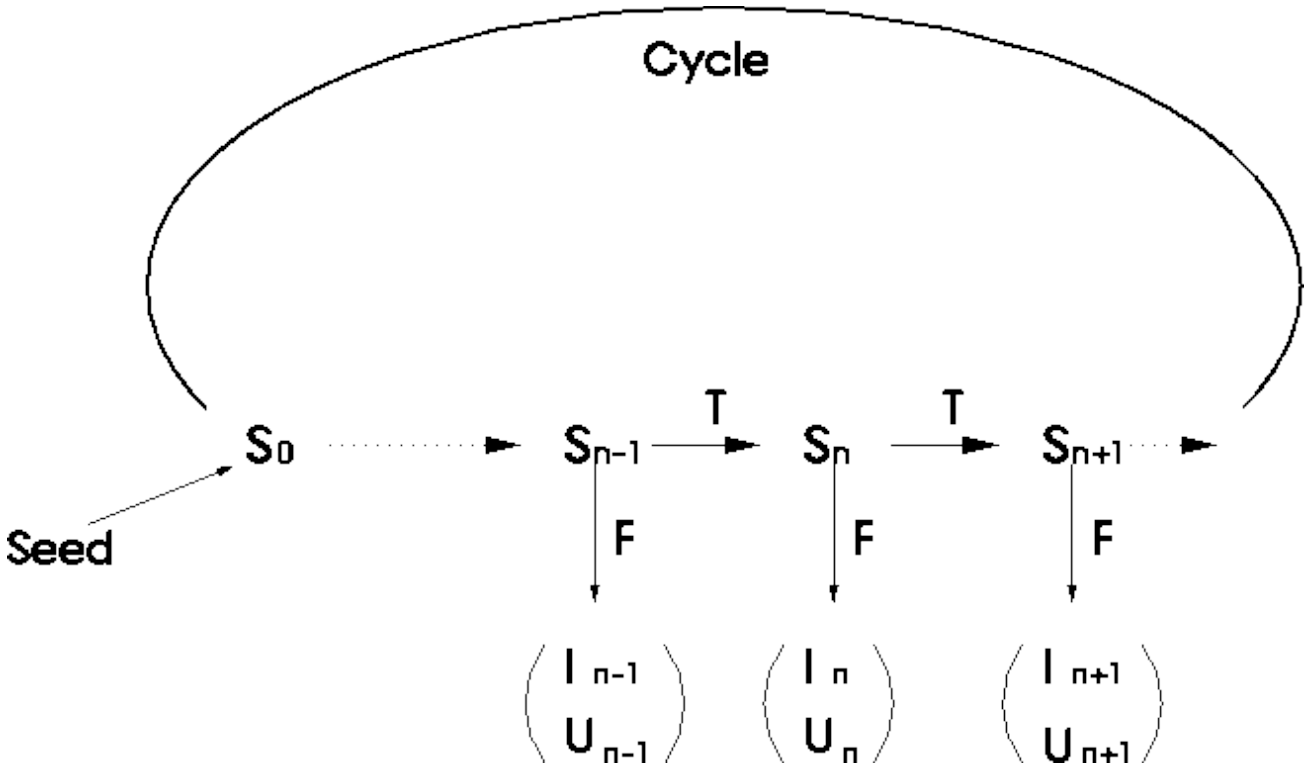
Desired Properties of RNG on a computer

- **Deterministic:** easy to debug (repeatable simulations).
- **Long Periods:** cycle length is long.
- **Uniformity:** RN generated in space evenly.
  - Go through complete cycle, all integers occur once!
  - TEST: N occurrences of x. The no. in (a,b) is equal to $N(b-a) \pm [N(b-a)]^{-1/2}$
- **Uncorrelated Sequences**:

  $\langle f_1(x_{i+1}) \ldots f_k(x_{i+k}) \rangle = \langle f_1(x_{i+1}) \rangle \ldots \langle f_k(x_{i+k}) \rangle$

  Monte Carlo can be very sensitive to such correlations, especially for large k.
- **Efficiency:** Needs to be fast.

Pseudo Random Sequence



S: State and initial seed.

T: Iteration process,

F: Mapping from state to integer RN (I) or real RN (U).

Period or Cycle Length (L)

- If internal state has M bits, total state space is 2^M values.
- If mapping is 1-1, space divides into a finite no. of cycles.
- Best case is a single cycle of length 2^M, otherwise $2^M/L$.
- It is easy to achieve very long cycle length: 32 or 24 bit generators are not long enough!
- Entire period of the RNG is exhausted in:

• rand (1 processor)	~	100 seconds
• drand48 (1 processor)	~	1 year
• drand48 (100 processor)	~	3 days
• SPRNG LFG (10^5 procs)	~	10^{375} years

Assuming 10^7 RN/sec per processor

e.g., In 1997, computer has ~10^8 RN/proc/sec (or 2^{26}).

Hence, it takes 1 sec to exhaust generator with 26 bits and 1 year for 2^{51} states.

e.g., consider 1024x1024 sites (spins) on a lattice. With MC, sweep 10^7 times to get averages.

You need to generate 10^{13} RN. A 32-bit RNG from 1980's dangerous on today's computers.

Common PRNG Generators

Linear Congruential Generator (LCG) **modulo (mod) is remainder after division**

64-bit word (or 2 32-bit LCG) give a period of ~10^{18}.

	Recurrence	Parallelization
• Multiplicative Lagged Fibonacci • Modified Lagged Fibonacci	$z_n = z_{n-k} * z_{n-l}$ $z_n = z_{n-k} + z_{n-l}$ (modulo 2^m)	vary initialization
• 48 bit LCG • 64 bit LCG • Prime Modulus LCG	$z_n = a*z_{n-1} + p$ (modulo m)	vary p
		vary a
• Combined Multiple Recursive	$z_n = a_{n-1}*z_{n-1} + \ldots + a_{n-k}*z_{n-k} + LCG$	vary LCG

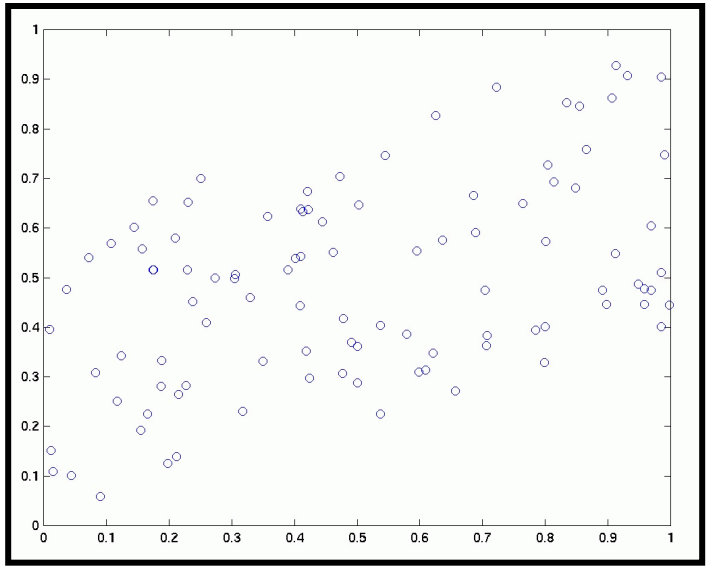
Uniformity

- Output consists of taking **N bits from state and making an integer in $(0,2^N-1)$ "I_k".**
- One gets a **uniform real "U_k" by multiplying by 2^{-N}**.
- We will study next time how to get other distributions.
- If there is a single cycle, then **integers must be uniform in the range $(0,2^N-1)$.**
- Uniformity of numbers taken 1 at a time is usually easy to guarantee. What we need is higher dimensional uniformity!

Sequential RNG Problems

- Correlations ⇒ non-uniformity in higher dimensions

Uniform in 1-D but
non-uniform in 2-D



This is the important property to guarantee:

$$< f_1(x_{i+1}) f_2(x_{i+2}) > = < f_1(x_{i+1}) >< f_2(x_{i+2}) >$$

MC uses numbers many at a time--they need to be uniform.

e.g., 128 atom MC moving 3N coords. at random needs 4 PRNG (x,y,z, i) or

128 x 4=1024. So every 1024 moves may be correlated!

Compare error for each PRNG, if similar OK. **(Use χ^2-test.)**

LCG Numbers fall in planes.

Good LCG generators have the planes close together.

For a k-bit generator, do not use them more than k/2 together.

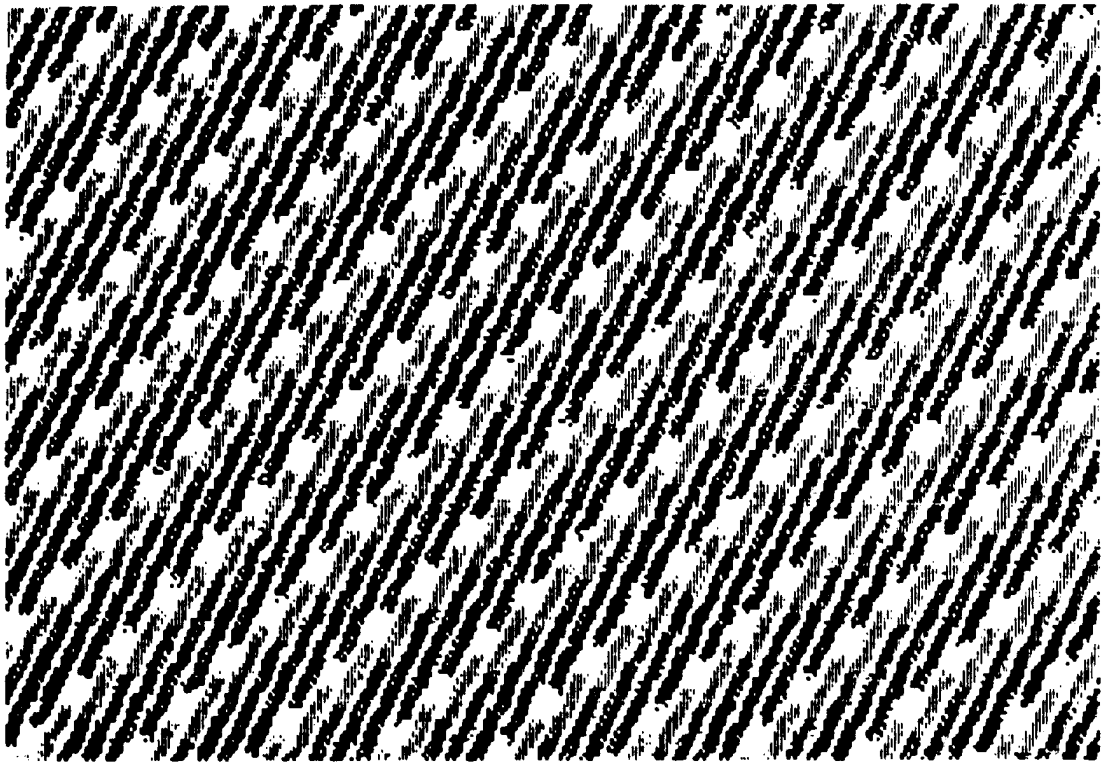


Fig. 8.2 Correlation between the triplets of points $\vec{r}_n$ $\{x_{3n}, x_{3n+1}, x_{3n+2}\}$ generated by a pseudorandom number generator. Th value of the third coordinate is represented by the intensity of the point.

What is a chisquare test?

- Suppose { x_1,x_2,….} are N Gaussian random numbers with mean zero, and variance 1
- What is the distribution of $y = \sum_{k=1}^{N} x^2$?
- $Q(\chi^2 \mid N)$ is the probability that y>χ^2
- The mean value of y is N, its variance is 2N.
- We use it to test hypotheses. Is the fluctuation we see, what we expect?

$$\chi^2 = \sum_{k=1}^{N} \frac{\left(x_k - x_k^{fit}\right)^2}{\sigma_k^2}$$

- Roughly speaking we should have $N - \sqrt{2N} < \chi^2 < N + \sqrt{2N}$
- More precisely look up value of $Q(\chi^2 \mid N)$ "chisquare function with N degrees of freedom"

Chi-squared test of randomness

- HW exercise: do test of several generators
  - Divide up "cube" into "N" bins.
  - Sample many "P" triplets.
  - Number/bin should be $n=P/N \pm (P/N)^{1/2}$

  In example: expected $n = 10 \pm (10)^{1/2}$

N=100

P=1000

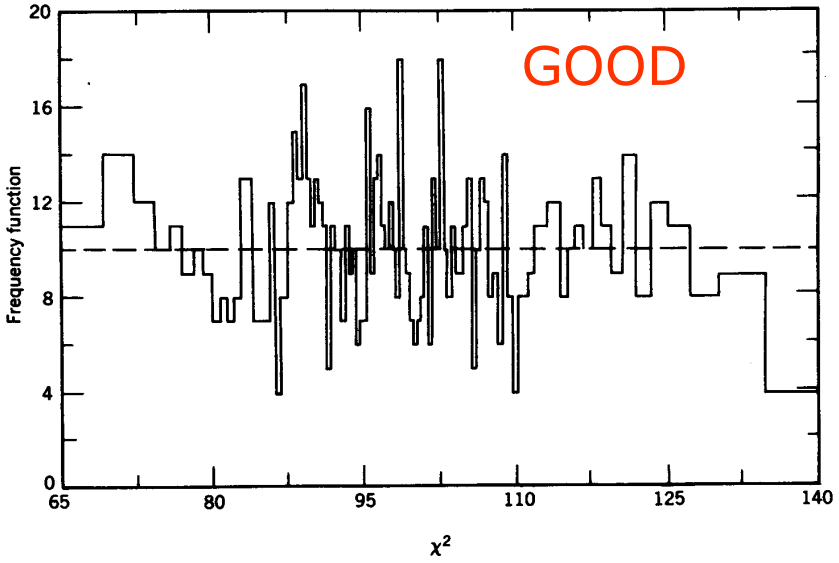


Figure A.1. The frequency function of observed values of χ^2 after 1000 samples. The bins are equally probable intervals where the expected number of χ^2 in each bin is 10. The pseudorandom number generator was Eq. (A.13).

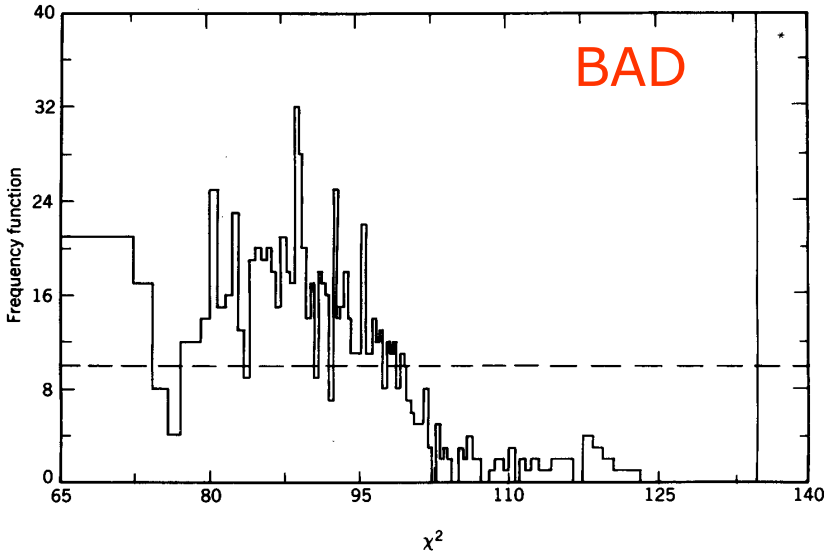


Figure A.2. The frequency function of observed values of χ^2 after 1000 samples using prn generator Eq. (A.14). The expected number of χ^2 in each bin is 10. The value of the bin indicated by (*) is offscale.

See "Numerical Recipes"

- **Chi-squared statistic** is $$\chi^2 = \sum_i \frac{(N_i - n_i)^2}{n_i}$$
  - **n_i (N_i) is the expected (observed) number in bin i.**
  - N_i is an integer! (Omit terms with $0 = n_i = N_i$)
  - Term with $n_i = 0$ and $N_i \neq 0$, correctly give $\chi^2 = \infty$.)
  - Large values of χ^2 indicate the "null" hypothesis, i.e. unlikely.
- The ***chi-squared probability*** is $Q(\chi^2|N-1)$
  (N-1) because of "sum rule".

The incomplete Γ-fct
N.R. Sec. 6.2 and 14.3

- Table may help with interpreting the ***chi-squared probability***.

Probability	Interpretation
< 1 %	**reject**
1 - 5 %	**suspect**
5 - 10 %	almost suspect
10 - 90 %	**pass**
90 - 95 %	almost suspect
95 - 99 %	**suspect**
> 99 %	**reject**

Recommendation:
For careful work use several generators!

- For most real-life MC simulations, *passing existing statistical tests is necessary but not sufficient*.
- Random number generators are still a ***black art***.
- ***Rerun with different generators***, since *algorithm may be sensitive to different RNG correlations*.
- **Computational effort is not wasted**, since results can be *combined to lower error bars*.

Parallel Simulations

- Parallel Monte Carlo is easy? Or is it?

- Two methods for easy parallel MC:
  - **Cloning**: same serial run, different random numbers.
  - **Scanning**: different physical parameters (density,…).

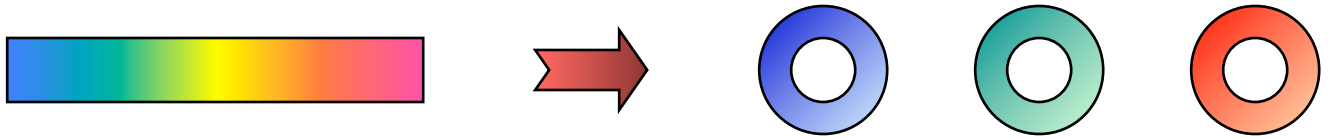
- Any parallel method (MPI, ..) can be used.

- Problems:
  - Big systems require excessive wall clock time.
  - Excessive amounts of output data generated.
  - Random number correlation?

Parallelization of RNGs

- Cycle Division: Leapfrog or Partition. Correlation problems

- Cycle Parameterization: If PRNG has several cycles, assign different cycles to each stream



Set of possible states

Each cycle gets a disjoint subset

- Iteration Parameterization: Assign a different iteration function to each stream

Examples of Parallel MC codes and Problems

- Embarrassingly parallel simulations. Each processor has its own simulation. Scanning and cloning.

  Unlikely to lead to problems unless they stay in phase.

- Lots of integrals in parallel (e.g. thousands of Feynman diagrams each to an accuracy of 10^{-6}).

  Problem if cycle length is exhausted.

- Particle splitting with new particles forking off new processes. Need lots of generators.

  Problem if generators are correlated initially.

- Space-time partitioning. Give each local region a processor and a generator.

  Problem if generators are correlated.