SU(2) and U(1) Symmetry

Isospin Conservation in Strong Interaction

In 1932, right after the discovery of the neutron, Heisenberg suggested that the similarity between proton and neutron mass implies that proton and neutron correspond to two degenerate states of strong interaction:

$$H\left|\psi_p\right\rangle = E\left|\psi_p\right\rangle$$
$$H\left|\psi_n\right\rangle = E\left|\psi_n\right\rangle$$

(note that mass (n) = 939.56 MeV and mass (p) = 938.27 MeV)

This degeneracy reminds us of the two-fold degeneracy of the spin-½ system for s_z = +½, s_z = -½, and we describe proton and neutron as two different states of the "nucleon", one with 'isospin' up and the other with isospin down.

$$\left|p\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$$
$$\left|n\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

A 'nucleon' state can be expressed as

$$\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} = \psi_p\begin{pmatrix}1\\ 0\end{pmatrix} + \psi_n\begin{pmatrix}0\\ 1\end{pmatrix}$$

and the transformation

$$\psi_p \to \psi'_p = \alpha\psi_p + \beta\psi_n$$
$$\psi_n \to \psi'_n = \gamma\psi_p + \delta\psi_n$$

would represent an equivalent state as far as strong interaction is concerned.

$$\begin{pmatrix} \psi'_p \\ \psi'_n \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = U \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$

The matrix U must be unitary to preserve the norm. Also, this implies $\left|\det\ U\right| = 1$.

Choosing det U = +1, the 2 x 2 unitary matrices form the SU(2) group. There are $2^2 - 1 = 3$ independent parameters.

For an infinitesimal transformation $U = 1 + i\xi$, where ξ is a 2 x 2 matrix whose elements are all small quantities. det U = 1 now implies Tr ξ = 0 and the condition that U be unitary, $(1 + i\xi)(1\text{-}i\xi^{+}) = 1$, implies $\xi = \xi^{+}$.

Thus ξ is a 2 x 2 traceless Hermitian matrix. It can be expressed as

$$\xi = \vec{\varepsilon} \bullet \vec{\tau}/2 \qquad \vec{\varepsilon} = \left(\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\right)$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\tau_i/2 , \tau_j/2 \right] = i\varepsilon_{ijk} \left(\tau_k/2 \right)$$

A finite transformation can be built from the infinitesimal transformation repeatedly, and one obtains

$$U = \exp\left(i\vec{\alpha} \bullet \vec{\tau}/2\right)$$

The invariance of the Hamiltonian H in SU(2) transformation leads to [H, U] = 0 and [H, $\vec{\tau}$] = 0. The eigenvalues of $\vec{\tau}$ are constants of the motion.

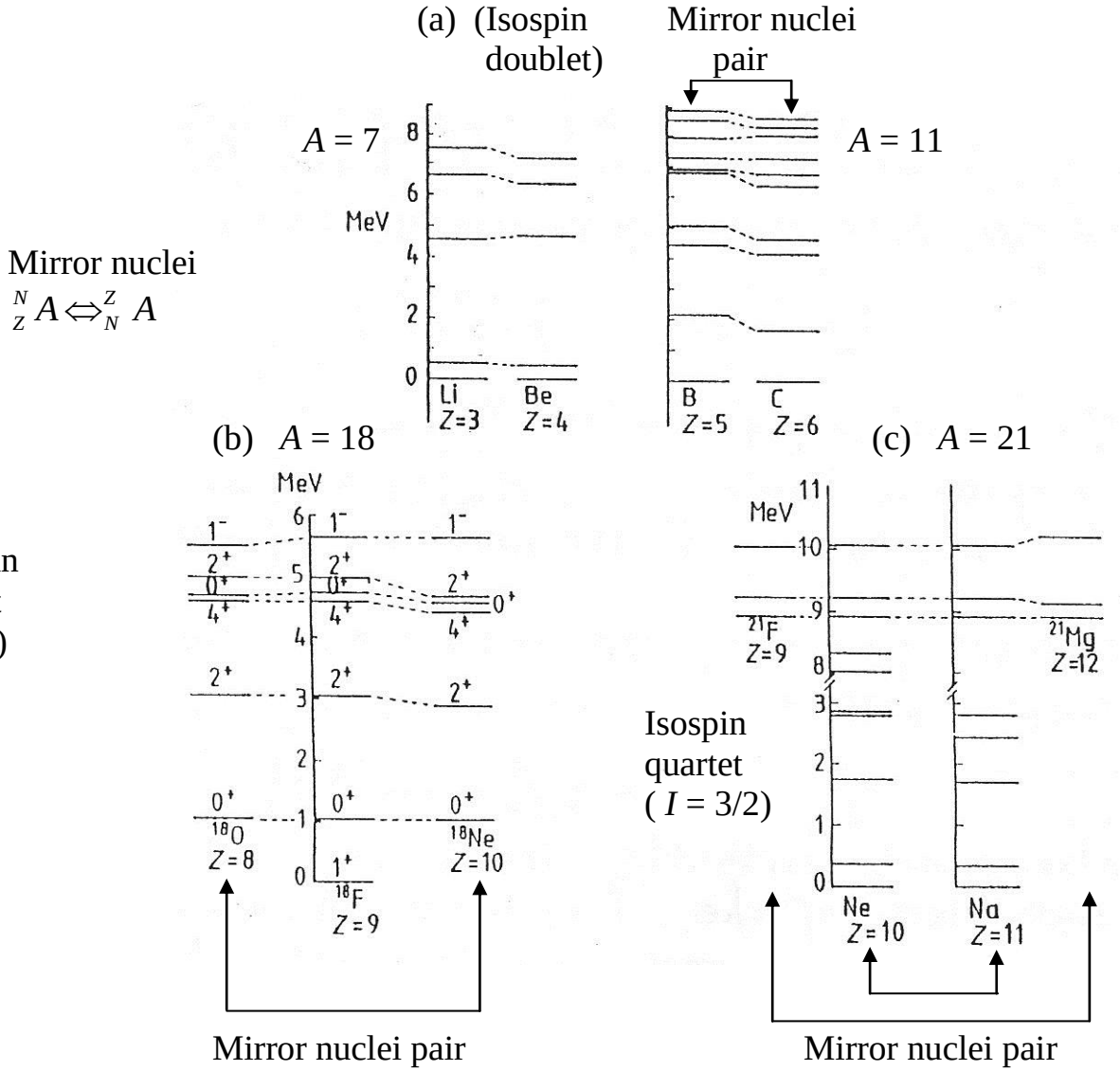
Consider states of several nucleons, the total isospin operators

$$\vec{T} = \frac{1}{2}\vec{\tau}_1 + \frac{1}{2}\vec{\tau}_2 + ...\frac{1}{2}\vec{\tau}_A$$

also commute with the Hamiltonian. Thus the eigenvalues of the $\vec{T}$ operators are constant of the motion. Energy levels of nuclei should be characterized by

eigenvalues of T^2 and T_3. Isospin should be a good quantum number. For each T, there are $2T + 1$ 'degenerate' states.

There are abundant examples for isospin conservation in nuclear physics. As shown in the figure below, the pairs of 'mirror nuclei' ${}^{7}_{3}\mathrm{Li}/{}^{7}_{4}\mathrm{Be}$ and ${}^{11}_{5}\mathrm{B}/{}^{11}_{6}\mathrm{C}$ have very similar binding energies for ground state as well as the excited states, after differences in Coulomb interaction (which does not conserve isospin) are corrected for. Similar isospin triplets and isospin quartets are observed for ${}^{18}_{8}\mathrm{O}/{}^{18}_{9}\mathrm{F}/{}^{18}_{10}\mathrm{Ne}$ system and for ${}^{21}_{9}\mathrm{F}/{}^{21}_{10}\mathrm{Ne}/{}^{21}_{11}\mathrm{Na}/{}^{21}_{12}\mathrm{Mg}$.



Isospin conservation also imposes important constraints on strong interaction processes. Some examples follow:

a) Consider the reaction $d + d \rightarrow {}^4\text{He} + \pi^o$

The isospins of deuteron and ^{4}He are both zero, while the isospin of π^o is 1. The initial state $d + d$ can only have total isospin 0, while the final state has total isospin 1. Therefore, this reaction violates isospin conservation and can proceed only via electromagnetic interaction. An extensive effort to measure the cross-section of this reaction led to a determination of 0.8 pb/sr as an upper limit (using 800 MeV beam). Very recently, however, experimenters at Indiana University claimed a successful detection of this reaction.

b) The $\psi' \rightarrow J\!/\!\psi + \pi^0 + \pi^0$ was known to occur with a branching ratio of ~ 20% (meaning ~ 20% of all ψ' decays end up in this channel). In contrast, $\psi' \rightarrow J\!/\!\psi + \pi^0$ has a much smaller branching ratio of 0.1%. Since $J\!/\!\psi$ and ψ' are $c\bar{c}$ bound states with isospin = 0, the $\psi' \rightarrow J\!/\!\psi + \pi^0$ decay violates isospin conservation. This explains why this decay mode is much inhibited compared with the $\psi' \rightarrow J\!/\!\psi + \pi^0 + \pi^0$ mode, which does not violate isospin conservation.

c) Consider the reactions

$$k^- + p \rightarrow \pi^0 + \Lambda \text{ and } k^- + n \rightarrow \pi^- + \Lambda$$

Λ has isospin 0, and $\pi + \Lambda$ can only have isospin = 1. $k^- + p$ can couple to $I = 0$ and $I = 1$, since k^- has $I = ½$, $I_z = -½$ and p has $I = ½$, $I_z = ½$.

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,0\rangle\right)$$

Similarly $k^- + n$ couples to $I = 1$ only

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = |1,-1\rangle$$

One therefore has

$$\sigma\left(k^- + p \to \pi^0 + \Lambda\right) \Big/ \sigma\left(k^- + n \to \pi^- + \Lambda\right) = 1/2$$

d) As another example similar to c), we consider the following reactions:

a:	$\pi^+ p \to \pi^+ p$	(elastic)
b:	$\pi^- p \to \pi^- p$	(elastic)
c:	$\pi^- p \to \pi^0 n$	(charge-exchange)

In these strong interactions, isospin is conserved. Now, consider the isospins of the following system:

$$\pi^+ p : |1,1\rangle \underbrace{\left|1/2, 1/2\right\rangle}_{\pi^+ \quad p} = \underbrace{\left|3/2, 3/2\right\rangle}_{\pi^+ p} \qquad \text{isospin}$$

where $|1,1\rangle$ signifies $|I, I_z\rangle$

$$\pi^- p : |1,-1\rangle \left|1/2, 1/2\right\rangle = \sqrt{\frac{1}{3}} \left|3/2, -1/2\right\rangle - \sqrt{\frac{2}{3}} \left|1/2, -1/2\right\rangle$$

$$\pi^0 n : |1,0\rangle \left|1/2, -1/2\right\rangle = \sqrt{\frac{2}{3}} \left|3/2, -1/2\right\rangle + \sqrt{\frac{1}{3}} \left|1/2, -1/2\right\rangle$$

Now the cross-section is proportional to $\left|\sum_I M_I\right|^2$, namely $\sigma \sim \left|\sum_I M_I\right|^2$

where

$$M_{1/2} = \left\langle \psi_f \left(I = 1/2\right) \middle| H \middle| \psi_i \left(I = 1/2\right) \right\rangle$$

$$M_{3/2} = \left\langle \psi_f \left(I = 3/2\right) \middle| H \middle| \psi_i \left(I = 3/2\right) \right\rangle$$

we obtain

$$\sigma_a : \sigma_b : \sigma_c = \left|M_{3/2}\right|^2 : \frac{1}{9}\left|M_{3/2} + 2M_{1/2}\right|^2 : \frac{2}{9}\left|M_{3/2} - 2M_{1/2}\right|^2$$

$M_{3/2} \gg M_{1/2}$; then $\sigma_a : \sigma_b : \sigma_c = 9 : 1 : 2$

$M_{1/2} \gg M_{3/2}$; then $\sigma_a : \sigma_b : \sigma_c = 0 : 2 : 1$

e) In addition to the (p, n) isospin doublet, there are many other examples of isospin multiplets for mesons and baryons:

$I = 3/2$	Δ^{++}	Δ^{+}	Δ^{0}	Δ^{-}
(quark content)	(uuu)	(uud)	(udd)	(ddd)
I_z	$3/2$	$1/2$	$-1/2$	$-3/2$
mass	1230.8 MeV	1231.6	1233.5	

$I = 1$	Σ^{+}	Σ^{0}	Σ^{-}
(quark content)	(uus)	(uds)	(dds)
I_z	1	0	-1
Mass	1189.4 MeV	1192.6	1197.4

$I = 1/2$	Ξ^{0}	Ξ^{1}
(quark content)	(uss)	(dss)
I_z	$+1/2$	$-1/2$
mass	1314.8 MeV	1322.3 MeV

I = 1 (mesons)	π^{+}	π^{0}	π^{-}
(quark content)	$\left(u\bar{d}\right)$	$\left(u\bar{u}/d\bar{d}\right)$	$\left(\bar{u}d\right)$
Iz	+1	0	-1
mass	139.57 MeV	134.98 MeV	139.57 MeV

f) Finally, remember that isospin is not conserved in electromagnetic interaction. Although *p*, *n* have very similar mass, their magnetic moments are very different:

$$\mu_n = -1.91\mu_N \qquad \mu_p = 2.79\mu_N$$

This difference reflects the electromagnetic origins for the magnetic moment.

Conservation of Charge, Baryon Number, and Other Additive Quantum Numbers

Isospin symmetry is an example of SU(2) unitary symmetry. Extension to SU(3) can describe strong interaction (quantum chromodynamics) which is based on the symmetry of 3 colors (red, blue, green).
There are also important implications when the unitary symmetry is applied to the transformation in one dimension. In the U(1) symmetry, the unitary $|x|$ transformation is

$$\psi' = e^{i\alpha G}\psi$$

Note that there is no SU(1) symmetry, since the constraint of $\det\left(e^{i\alpha G}\right)=1$ would completely fix the transformation to a trivial $\psi' = \psi$ transformation.

G is the generator of the U(1) transformation, and *G* is a conserved observable if the Hamiltonian *H* commutes with the U(1) transformation. Therefore, the energy eigenstate can also be an eigenstate for the generator *G*:

$$G\psi = q\psi \qquad H\psi = E\psi$$

The eigenvalue of *G* can be identified as the charge of a particle, for example. Charge conservation is therefore a consequence of U(1) symmetry.

Identical algebra can be adopted to describe other additive conservation laws such as baryons number conservation, Lepton number conservation, strangeness conservation, etc. These conservation laws are derived based on the assumption that [*H*, *G*] = 0. Whether this is indeed true can only be tested by experiments. The charge conservation, however, is regarded as more robust compared with the other conservation laws (such as baryon number) since a dynamic theory, quantum electrodynamics, can be deduced based on the local gauge symmetry of U(1).

Just like the isospin symmetry in SU(2), the charge conservation in U(1) gives rise to additive quantum numbers. Consider a system consisting of *n* particles, the corresponding U(1) transformation is

$$e^{i\alpha G_1} e^{i\alpha G_2} \ldots\ldots e^{i\alpha G_n}$$
$$= e^{i\alpha (G_1 + G_2 + \ldots\ldots + G_n)}$$

Hence, $G = G_1 + G_2 + \ldots\ldots + G_n$ is a conserved observable.

We now consider baryon number conservation. This conservation law is motivated by the fact that all known reactions and decay processes involving baryons conserve baryon numbers. In particular, the lightest baryon, proton, appears to be perfectly stable.

We mentioned already that charge conservation is a result of a field (electromagnetic field) coupled to the electric charge. Lee and Yang suggested in 1955 that the apparent conservation of baryon numbers implies the existence of a long-range field coupled to baryon number (analogous to the case for charge conservation ↔ electromagnetic field). In particular, the gravitational force between an object and the earth

$$F_{\text{Gravity}} = K \frac{M_G M_{\text{earth}}}{r^2}$$

could contain an additional term sensitive to the total baryon number of the object

$$F_{\text{Baryon}} = K_B \frac{\left(M_N B\right)\left(M_N B_{\text{earth}}\right)}{r^2}$$

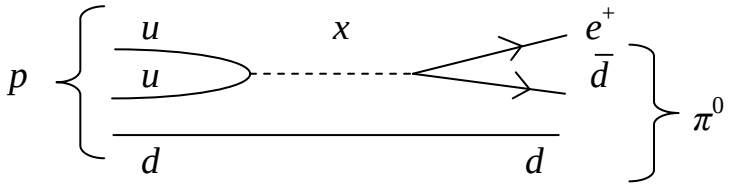
To measure such new form of force coupled to baryon number (the fifth force), one could compare the gravitational force of two different objects having the same mass, but different total number of baryons. This is possible by selecting two objects made of different materials. The difference in the nuclear binding energies would give different total baryon number for equal mass. By a comparison of objects made of aluminum and platinum, it was found that

$$K_B < 10^{-9}\, K$$

Altough the standard model does not allow proton decay (none of the fermion-fermion-boson coupling diagrams we introduced in Chapter 1 allows proton decay), explanation for a well-known phenomenon in cosmology, namely the matter-antimatter asymmetry, requires baryon number non-conservation (the Sakharov conditions).

In Grand Unified Theories (GUT), such as SU(5), there exist "leptoquark" with mass $\simeq 10^{15}$ GeV. The leptoquark can couple to qq or ℓq.

A possible diagram for proton decay is



The most favorable decay channel in SU(5) is $p \rightarrow e^{+}\pi^{0}$. In supersymmetry GUT, the most favored decay channel is $p \rightarrow \bar{\nu}_{\tau} k^{+}$. Note that in these decays both baryon number conservation and lepton number conservation are violated. However, $B - L$, the difference between baryon number and lepton number, is conserved.