Lecture 6/7 (February 10/12, 2014)

DIRAC EQUATION

The non-relativistic Schrödinger equation was obtained by noting that the Hamiltonian

$$H = P^2 \big/ 2m \tag{1}$$

can be transformed into an operator form with the substitutions

$$H \to i\frac{\partial}{\partial t} \quad (\hbar = 1,\ c = 1) \tag{2}$$
$$P \to -i\vec{\nabla}$$

resulting
$$i\frac{\partial}{\partial t}\psi(x,\ t) = \frac{\nabla^2}{2m}\psi(x,\ t) \tag{3}$$

This equation is linear in the time-derivative and quadratic in the space derivative, and is manifestly non-covariant under Lorentz transformation. Clearly, the left-hand side and the right-hand side of Equation 3 transforms differently under Lorentz transformation.

We briefly review Lorentz transformation.

Space-time coordinates (t, x, y, z) are denoted by the contravariant 4-vector

$$x^\mu \equiv \left(x^o,\ x^1,\ x^2,\ x^3\right) \equiv (t,\ x,\ y,\ z\ )$$

The covariant 4-vector, x_μ, is

$$x_\mu \equiv \left(x_o,\ x_1,\ x_2,\ x_3\right) \equiv (t,\ \text{-}x,\ \text{-}y,\ \text{-}z) = g_{\mu\nu}x^\nu$$
$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4}$$

$g_{\mu\upsilon}$ is the metric tensor

The Lorentz transformation relates $x^{\mu\prime}$ to x^{μ}:

$$\left(x^{\mu}\right)' = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \tag{5}$$

For an inertial system S' moving along z with $v/c = \beta$, we have

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \tag{6}$$

where $\gamma = \left(1-\beta^2\right)^{-1/2}$

Any set of four quantities which transform like x^{μ} under Lorentz transformation is called a four-vector. The total energy E and the momentum $\vec{P}$ form a four-vector

$$P^{\mu} = \left(P^{o},\ P^{1},\ P^{2},\ P^{3}\right) = \left(\mathrm{E},\ \vec{P}\right) \tag{7}$$

Scalar product of two 4-vectors are invariant under Lorentz transformation

$$A_{\mu}B^{\mu} = g_{\mu\nu}A^{\nu}B^{\mu} \tag{8}$$

Some examples:

$$x\bullet x = x_{\mu}x^{\mu} = t^2 - x^2$$
$$P\bullet P = P_{\mu}P^{\mu} = \mathrm{E}^2 - P^2$$
$$x\bullet P = x_{\mu}P^{\mu} = t\ \mathrm{E} - \vec{x}\bullet\vec{P}$$

The $H \to i\dfrac{\partial}{\partial t},\ \vec{P} \to -i\vec{\nabla}$ transcription is Lorentz covariant

$$P^{\mu} \to i\frac{\partial}{\partial x_{\mu}} = i\partial^{\mu}$$

One can readily obtain the continuity equation from the Schrödinger equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \bullet \vec{J} = 0 \tag{9}$$

where $\rho = \psi^* \psi \geq 0,\ \vec{J} = -\frac{i}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$

ρ is the probability density, and $\vec{J}$ is the current density.

Note that the continuity equation can be written in a covariant form

$$\partial_\mu j^\mu = 0, \quad j^\mu = \left(\rho,\ \vec{J}\right) \tag{10}$$

The early attempt for formulating a Lorentz covariant equation started from the relativistic energy-momentum relation

$$\mathrm{E}^2 = P^2 + m^2$$

with the $\mathrm{E} \to i\frac{\partial}{\partial t},\ \vec{P} \to -i\vec{\nabla}$ transcription, one obtains

$$-\frac{\partial^2}{\partial t^2}\psi = \left(-\nabla^2 + m^2\right)\psi \tag{11}$$

which is the relativistic Schrödinger equation, or the Klein-Gordon equation.

Equation 11 can be written as

$$\left(\Box^2 + m^2\right)\psi = 0 \tag{12}$$

where

$$\Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial_\mu \partial^\mu \tag{13}$$

is the d'Alembertian operator, a Lorentz invariant operator.

From the Klein-Gordon equation, one can readily derive the following continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \bullet \vec{J} = 0$$

$$\left.\begin{aligned} \rho &= i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right) \\ \vec{J} &= -i\left(\phi^* \vec{\nabla}\phi - \phi \vec{\nabla}\phi^*\right) \end{aligned}\right\} j^\mu = i\left(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*\right) \qquad (14)\ (15)$$

For a plane-wave solution to Klein-Gordon equation

$$\phi = Ne^{-i\left(\mathrm{E}t - \vec{P}\bullet\vec{x}\right)} \qquad (16)$$

We have

$$\rho = 2E\left|N\right|^2, \quad \vec{J} = 2\vec{P}\left|N\right|^2 \qquad (17)$$

Since $\mathrm{E} = \pm\left(P^2 + m^2\right)^{1/2}$, ρ can be negative for the E < 0 solutions.

The origin of this problem is the second derivative in time for the term $\partial^2/\partial t^2$.

Dirac tried to find a relativistic covariant equation with a positive-definite probability density. He assumed an equation linear in $\frac{\partial}{\partial t}$ and in $\vec{\nabla}$:

$$\begin{aligned} H\psi &= \left(\vec{\alpha}\bullet\vec{P} + \beta m\right)\psi \\ \vec{\alpha}\bullet\vec{P} &= \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 \end{aligned} \qquad (18)$$

It is evident that α's and β cannot be just numbers, because the equation would not be invariant even under spatial rotation. Dirac proposed that the equation be considered as a matrix equation. It should satisfy $H^2\psi = \left(P^2 + m^2\right)\psi$.

Applying H to Equation 18 leads to

$$H^2\psi = \left[\sum_i \alpha_i^2 P_i^2 + \sum_{i\neq j}\left(\alpha_i\alpha_j + \alpha_j\alpha_i\right)P_iP_j + \sum_i\left(\alpha_i\beta + \beta\alpha_i\right)mP_i + \beta^2 m^2\right]\psi$$

Therefore

$$\{\alpha_i,\ \alpha_j\} = 2\delta_{ij}\ ;\ \{\alpha_i,\ \beta\} = 0$$
$$\alpha_i^2 = \beta^2 = 1$$

What are the other properties of α_i, β?
First, α_i, β are Hermitian, since H is Hermitian (see Equation 18).

In addition, $Tr(\alpha_i) = 0$, $Tr(\beta) = 0$. This can be shown by the anticommutator relations in Equation 19. Recall $Tr(AB) = Tr(BA)$

$$\begin{aligned} Tr(\alpha_i) &= Tr(\alpha_i\beta^2) = Tr(\beta\alpha_i\beta) = Tr(-\alpha_i\beta\beta) \\ &= -Tr(\alpha_i) \Rightarrow Tr(\alpha_i) = 0 \end{aligned} \tag{20}$$

Similarly, $Tr(\beta) = 0$. Therefore, α_i, β are traceless, Hermitian matrices.

The eigenvalues of α_i, β can only be +1 or -1 since $\alpha_i^2 = \beta^2 = 1$.

$$\alpha_i x = \lambda x \ \Rightarrow\ \alpha_i^2 x = \lambda^2 x = x \ \Rightarrow\ \lambda = \pm 1 \tag{21}$$

The trace of a matrix is equal to the sum of all its eigenvalues. Therefore, for a traceless matrix having +1, -1 as possible eigenvalues, it must have an even number of dimension n. n cannot be 2, since only three traceless $n = 2$ independent matrices exist $(\sigma_x,\ \sigma_y,\ \sigma_z,$ for example$)$, and we need four traceless matrices to accommodate $\alpha_1, \alpha_2, \alpha_3, \beta$.

$n = 4$ is the smallest possible dimension for α_i, β. The Pauli-Dirac representation is given as

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{22}$$

The Dirac equation $\left[\vec{\alpha}\bullet\vec{P} + m\beta\right]\psi = H\psi$ can be expressed in a more covariant fashion by multiplying β from the left:

$$\left(\beta H - (\beta\vec{\alpha})\bullet\vec{P} - m\right)\psi = 0 \tag{23}$$

Introducing $\gamma^\mu = (\beta,\ \beta\vec{\alpha})$, Equation 23 becomes

$$\left(\gamma^\mu P_\mu - m\right)\psi = 0 \tag{24}$$

$P_\mu = i\partial_\mu$, therefore

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \qquad \left(\text{recall } \partial_\mu = \frac{\partial}{\partial x^\mu}\right) \tag{25}$$

Explicitly,

$$\gamma^o = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{26}$$

γ^μ's satisfy the anticommutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I \tag{27}$$
$$\left(\gamma^o\right)^2 = I, \qquad \left(\gamma^i\right)^2 = -I$$

Although γ^o is Hermitian, $(\gamma^o)^+ = \gamma^o$, γ^i is anti-Hermitian:

$$(\gamma^i)^+ = -\gamma^i \tag{28}$$

However,

$$(\gamma^\mu)^+ = \gamma^o \gamma^\mu \gamma^o \qquad \text{and} \qquad \gamma^\mu = \gamma^o (\gamma^\mu)^+ \gamma^o \tag{29}$$

To obtain the adjoint Dirac equation, one takes Hermitian conjugate of Equation 25

$$-i\partial_\mu \psi^+ \left(\gamma^\mu\right)^+ - m\psi^+ = 0 \tag{30}$$

Since γ^μ is not Hermitian for μ = 1, 2, 3, it is useful to multiply Equation 30 from the right by γ^o to yield (using Equation 29)

$$-i\partial_\mu \left(\psi^+ \gamma^o\right)\gamma^\mu - m\left(\psi^+ \gamma^o\right) = 0 \tag{31}$$

Define $$\bar{\psi} = \psi^+ \gamma^o \tag{32}$$

the adjoint Dirac equation is $$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{33}$$

or $$\bar{\psi}\left(\not{P} + m\right) = 0 \tag{34}$$

where $$\not{P} = \gamma^{\mu} P_{\mu}$$

and *P* acts to its left in Equation 34.

One is now ready to check the continuity equation and the probability density and current density for the Dirac equation.

Multiplying Equation 25 by $\bar{\psi}$ to its left, and Equation 33 by ψ to its right, we obtain

$$\begin{aligned} i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \bar{\psi}m\psi = 0 \\ i\partial_{\mu}\bar{\psi}\gamma^{\mu}\psi + m\bar{\psi}\psi = 0 \end{aligned} \tag{35}$$

Adding these two equations yields

$$\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\psi\right) = 0 \tag{36}$$

Define $$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi \tag{37}$$

the continuity equation $\partial_{\mu} j^{\mu} = 0$ follows.

Note that $$\rho = j^{o} = \bar{\psi}\gamma^{o}\psi = \psi^{+}\psi = \sum_{i=1}^{4} \left|\psi_i\right|^2 \geq 0$$

Hence the probability density is indeed positive definite for the Dirac equation.

As will be shown later j^{μ} transforms like a 4-vector, and $\rho = j^{o}$ is not a Lorentz invariant. Rather, it is the time component of a 4-vector. It is not too surprising that ρ is not a Lorentz invariant, since a probability density satisfies the following:

$$\int \rho d^3x = \text{const} \tag{38}$$

Since d^3x undergoes Lorentz contraction, it is natural that ρ must also change under the Lorentz transformation in order to preserve the integral.