Solution to Homework 3

Physics 485: Fall 2017

Problem I. QUANTUM DOT AND PARTICLE IN INFINITE SQUARE WELL

1. The time-independent Schrödinger equation is

\[
\begin{align*}
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} &= E\psi \quad 0 < x < L \\
\psi &= 0 \quad \text{otherwise}
\end{align*}
\]

We can solve it and get

\[
\psi = A\sin\left(\frac{2mE}{\hbar^2}x\right) + B\cos\left(\frac{2mE}{\hbar^2}x\right)
\]

And we know the wavefunction should be continuous, so \(\psi(0) = \psi(L) = 0\). Then we get

\[
E_n = \frac{\hbar^2}{2m} \left(\frac{n\pi}{L}\right)^2
\]

\[
\psi = A\sin(n\pi x/L)
\]

\(A\) is a constant for normalization. After some math we can know it is \(\sqrt{2/L}\).

2. From the result of previous question. We know \(E = \frac{\hbar^2}{2m} \left(\frac{n\pi}{L}\right)^2\). Plus the first 3 levels are corresponding to \(n = 1, 2, 3\), substitute \(L = 2.5nm\), we can finally get

\[
E_1 = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 \approx 9.64 \times 10^{-21} J = 0.06eV
\]

\[
E_2 = \frac{\hbar^2}{2m} \left(\frac{2\pi}{L}\right)^2 \approx 3.86 \times 10^{-20} J = 0.24eV
\]

\[
E_3 = \frac{\hbar^2}{2m} \left(\frac{3\pi}{L}\right)^2 \approx 8.68 \times 10^{-20} J = 0.54eV
\]

3. If the electron jump from its fifth excited state to its ground state and emits a photon, the energy of the photon should be

\[
E_p = E_6 - E_1 = (36 - 1) \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 \approx 3.37 \times 10^{-19} J
\]

And from \(E_p = h\nu = h\nu/\lambda\) we can get

\[
\lambda = \frac{h\nu}{E_p} \approx 5.89 \times 10^{-7} m = 589nm
\]

It is in the visible range.

Problem II. INFINITE SQUARE WELL II
1. \begin{equation}
\int |\psi|^2 dx = \int dx \left( \frac{1}{4} |\psi_1|^2 + \frac{3}{4} |\psi_2|^2 + \frac{i\sqrt{3}}{4} \psi_1^* \psi_2 + \frac{-i\sqrt{3}}{4} \psi_2^* \psi_1 \right)
\end{equation}
Because \( \psi_1, \psi_2 \) are eigenstates of different energy, they are orthogonal to each other. Plus \( \psi_1, \psi_2 \) are normalized, we can evaluate equation (7) as
\begin{equation}
\int |\psi|^2 dx = \frac{1}{4} + \frac{3}{4} = 1
\end{equation}
Therefore, \( \psi \) is normalized.

2. The probability amplitude for \( \psi_1 \) is \( i = \frac{1}{\sqrt{2}} \), and the probability amplitude for \( \psi_2 \) is \( p_3 = \frac{1}{\sqrt{2}} \). Therefore, the probability a measurement of energy yields \( E_1 \) is
\begin{equation}
P_1 = \frac{i}{2} \times \frac{-i}{2} = \frac{1}{4}
\end{equation}
And the probability a measurement of energy yields \( E_2 \) is
\begin{equation}
P_2 = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{4}
\end{equation}

3. The expectation value of energy is
\begin{equation}
\langle E \rangle = P_1 E_1 + P_2 E_2 = \frac{1}{4} E_1 + \frac{3}{4} E_2
\end{equation}
Next, the standard deviation of energy is
\begin{equation}
\Delta E = \sqrt{P_1 (E_1 - \langle E \rangle)^2 + P_2 (E_2 - \langle E \rangle)^2} = \frac{\sqrt{3}}{4} |E_1 - E_2|
\end{equation}

4. From the time-dependent Schrodinger equation \( i\hbar \partial_t \psi = H \psi \), we know that if \( \psi(x,t=0) = \psi_n \) is the eigenwavefunction of energy \( E_n \) then \( \psi(x,t) = \exp(-iE_n t/\hbar)\psi_n \). Therefore, we can get
\begin{equation}
\psi(x,t) = \frac{i}{2} \exp(-i E_1 t/\hbar) \psi_1 + \frac{\sqrt{3}}{2} \exp(-i E_2 t/\hbar) \psi_2
\end{equation}
And then \( |\psi(x,t)|^2 \) should be
\begin{equation}
|\psi(x,t)|^2 = \frac{1}{4} |\psi_1|^2 + \frac{3}{4} |\psi_2|^2 - \frac{i\sqrt{3}}{4} \exp(-i \frac{E_2 - E_1}{\hbar} t) \psi_1^* \psi_2 + \frac{\sqrt{3}}{4} \exp(i \frac{E_2 - E_1}{\hbar} t) \psi_2^* \psi_1
\end{equation}
Substitute \( \psi_1 = \sqrt{2/L} \sin \pi x / L \), \( \psi_2 = \sqrt{2/L} \sin 2\pi x / L \) and \( E_n = \frac{\hbar^2}{2m} (\frac{\pi n}{L})^2 \) into equation (14)
\begin{equation}
|\psi(x,t)|^2 = \frac{1}{2L} \sin^2 \frac{\pi x}{L} + \frac{3}{2L} \sin^2 \frac{2\pi x}{L} - \frac{\sqrt{3}}{L} \sin \left( \frac{3\hbar \pi^2 t}{2mL^2} \right) \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L}
\end{equation}

5. When \( t = 0 \) and \( t = \frac{2mL^2}{3\hbar} \), the second term vanishes because \( \sin(0) = \sin(\pi) = 0 \), so
\begin{equation}
|\psi|^2 = \frac{1}{2L} \sin^2 \frac{\pi x}{L} + \frac{3}{2L} \sin^2 \frac{2\pi x}{L}
\end{equation}
And the plot of it is shown in figure 1.
\[ \langle x(t) \rangle = \int_0^L dx \, x |\psi(x, t)|^2 \]
\[ = \int_0^L dx \, x \frac{1}{2L} \sin^2 \frac{\pi x}{L} + \int_0^L dx \, x \frac{3}{2L} \sin^2 \frac{2\pi x}{L} - \int_0^L dx \, \frac{\sqrt{3}}{L} \sin \left( \frac{3\pi^2}{2mL^2} t \right) \frac{\pi x}{L} \sin \frac{2\pi x}{L} \]
\[ = \frac{L}{8} + \frac{3L}{8} + \frac{8\sqrt{3}L}{9\pi^2} \sin \left( \frac{3\pi^2}{2mL^2} t \right) \]
\[ = \frac{L}{2} + \frac{8\sqrt{3}L}{9\pi^2} \sin \left( \frac{3\pi^2}{2mL^2} t \right) \]  
(17)

It oscillates around \( x = L/2 \) with amplitude \( \frac{8\sqrt{3}}{9\pi^2} \) and frequency \( \frac{3\pi^2}{2mL^2} \).

**Problem III. DECOMPOSING WAVEFUNCTIONS**

1. \[ 1 = \int_{-L/2}^{L/2} dx \, |\psi(x)|^2 = A^2 \int_{-L/2}^{L/2} dx \, (x^4 - \frac{L^2}{2} x^2 + \frac{L^4}{16}) \]
\[ = A^2 \frac{L^5}{30} \]  
(18)

Consequently, we can choose \( A = \sqrt{30/L^3} \) to normalize the wave function.

2. As stated in the question, the eigenwavefunction of this ISW is
\[ \psi_n(x) = \begin{cases} 
\frac{\sqrt{2}/L \cos \frac{n\pi x}{L}}{\sqrt{30/L^3}} & n \text{ odd} \\
\frac{\sqrt{2}/L \sin \frac{n\pi x}{L}}{\sqrt{30/L^3}} & n \text{ even} 
\end{cases} \]  
(19)

Now we want to get the \( a_1 \) in \( \psi(x) = \sum_{n=1}^{\infty} a_n \psi_n(x) \)
\[ a_1 = \int_{-L/2}^{L/2} dx \, \psi^*_1 \psi(x) = \int_{-L/2}^{L/2} dx \, \left( \frac{\sqrt{2}/L \cos \frac{\pi x}{L}}{\sqrt{30/L^3}} \right) \frac{\sqrt{30/L^3} (x^2 - L^2/4)}{\pi^3} \]
\[ = \frac{-8\sqrt{15}}{\pi^3} \]  
(20)

Then \( P_1 = |a_1|^2 \approx 0.9986 \).
3. Similarly, we can get

$$a_2 = \int_{-L/2}^{L/2} dx \; \psi_2^* \psi_2(x) = \int_{-L/2}^{L/2} dx \; \left( \sqrt{2/L} \sin \frac{2\pi x}{L} \right) \left( \sqrt{30/L^5} (x^2 - L^2/4) \right) = 0$$

(21)

Therefore, \( P_2 = |a_2|^2 = 0 \)

4. As what example 3.5 in Townsend’s book do,

$$\langle E \rangle = -\frac{\hbar^2}{2m} \int_{-L/2}^{L/2} dx \sqrt{\frac{30}{L^5} (x^2 - \frac{L^2}{4})} \left( \sqrt{\frac{30}{L^5} (x^2 - \frac{L^2}{4})} \right)$$

$$= \frac{5\hbar^2}{mL^2} \geq E_1 = \frac{\pi^2 \hbar^2}{2mL^2} \approx 4.93 \frac{\hbar^2}{mL^2}$$

(22)

**Problem IV. PARTICLE IN A DOUBLE POTENTIAL WELL**

1. The plots are shown in figure 2.

2. What I draw in question 1 are ground state and second excited state respectively. For \( E_c \), it is not a bounded state, so we cannot judge which level it is in from the nodes (because the nodes is infinity). Reviewing the ISW situation, when the energy level increases by 1 the nodes increases by 1, for \( E_B \) the system can be regarded as a ’distorted’ well, so when the energy level increases by 1 the nodes increases by 1. Higher energy leads to higher frequency and then more nodes. Therefore from the number of nodes for those 3 energy in figure 2, we can see \( N_C > N_B > N_A \) which is consistent with \( E_C > E_B > E_A \).

3. We know that \( |y''| \propto k^2 A \). From equation (4.43) in Townsend’s book, we know that \( k^2 A \sim \sqrt{k^4 \psi^2 + k^2 (\frac{d\psi}{dx})^2} \), and \( \psi, \frac{d\psi}{dx} \) are continuous, so the greatest curvature corresponds to greatest \( k \). Consequently, for three plots in figure 2, the greatest curvature can be found at the stationary point in region 2 and 4.

4. From equation (4.43) in Townsend’s book, we know that the greatest amplitude corresponds to smallest \( k \). Therefore, in the plot of \( E_A \), the greatest amplitude can be found in region 2 and 4; in the plot of \( E_B \), the greatest amplitude can be found in region 3; in the plot of \( E_C \), the greatest amplitude can be found in region 1 and 5.

5. When \( V > E \), the wavefunction become exponential. It does not agree to the classical mechanics, since in classical mechanics, particles cannot go into the region of \( V > E \). We can find this category in the wavefunctions of \( E_A \) and \( E_B \).
Figure 2: Plots of wavefunctions of 3 different energy