I. TRANSMISSION THROUGH A RECTANGULAR BARRIER \[8\]

To calculate the transmission coefficients for the three given barrier widths in this problem, we can use the following equation

\[
\log(T) \approx -2 \sqrt{\frac{2m(V - E)}{\hbar^2} L}.
\]

Here, \(V\) and \(L\) are the height and the width of the rectangular barrier, respectively.

(a) For \(L_a = 10^{-10}\) m Eq. (1) gives

\[
\log(T_1) \approx -2 \sqrt{\frac{2 \times 9.01 \times 10^{-31} \text{kg} \times 2 \times 1.60 \times 10^{-19} \text{J}}{(1.05 \times 10^{-34} \text{J} \cdot \text{s})^2}} \times 10^{-10} \text{m} \approx 1.45 \Rightarrow T_a \approx 0.235
\]

(b) Performing the same calculation as in part (a) for \(L_b = 9 \times 10^{-9}\) yields \(T_b \approx 1.28 \times 10^{-57} \approx 0\).

(c) Performing the same calculation as in part (a) for \(L_c = 10^{-9}\) gives \(T_c \approx 5.04 \times 10^{-7}\).

(d) Considering Eq. (1) we see that \(T \propto e^{-L}\) so that the quantum mechanical prediction is that increasing the width of the barrier results in an exponential decrease of \(T\). From calculations above, \(T_a > T_c > T_b\), which is consistent with \(L_c > L_a > L_b\).

Additionally, noting that \(L_b = 9 \times 10^{-9}\) m is almost the same width as approximately 10 atoms, it is reasonable to expect that it is nearly impossible for an electron to tunnel through a barrier having this width. Consequently, finding that \(T_b\) is rather small is consistent with quantum physics expectations.

II. TWO-DIMENSIONAL INFINITE SQUARE WELL \[13\]

We consider a particle of mass \(m\) in a two-dimensional infinite square well potential having sides of length \(a\) and \(b\) given by

\[
V(x) = \begin{cases} 
0 & 0 < x < a \text{ and } 0 < y < b \\
\infty & \text{otherwise}
\end{cases}
\]

(a) Recognizing that this potential is “two copies” of a one-dimensional infinite square well, we can form its wavefunctions as simple products of the one-dimensional wavefunctions

\[
\psi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{a} \right).
\]

(Note that this statement is simply an informal way of saying that the two-dimensional Schrodinger equation can be solved by separation of variables, as in Sec. 6.1 of Townsend.) More precisely, for the two-dimensional infinite square well we can write eigenfunctions as

\[
\psi_{n_x, n_y} = \frac{2}{a} \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{a} \right)
\]

with corresponding energy eigenvalues given by

\[
E_{n_x, n_y} = \frac{\hbar^2}{2m} \left( \frac{n_x \pi}{a} \right)^2 + \left( \frac{n_y \pi}{b} \right)^2.
\]
(b) Labeling the first eight states by \((n_x, n_y)\) we find

\begin{align*}
(1, 1) \rightarrow & E_1 = \frac{\hbar^2}{m} \left( \frac{\pi}{a} \right)^2 \\
(2, 1) \ (1, 2) \rightarrow & E_2 = \frac{5\hbar^2}{2m} \left( \frac{\pi}{a} \right)^2 \\
(2, 2) \rightarrow & E_3 = \frac{4\hbar^2}{m} \left( \frac{\pi}{a} \right)^2 \\
(3, 1) \ (1, 3) \rightarrow & E_4 = \frac{5\hbar^2}{m} \left( \frac{\pi}{a} \right)^2 \\
(3, 2) \ (2, 3) \rightarrow & E_5 = \frac{13\hbar^2}{2m} \left( \frac{\pi}{a} \right)^2.
\end{align*}

As indicated in Eq. (6) some of these states have the same energy (for example states described by \((1, 2)\) and \((2, 1)\) both correspond to an energy given by \(E_2\)). While \(E_1\) and \(E_3\) are non-degenerate (i.e. they correspond to a single wavefunction), energy eigenvalues \(E_2, E_4\) and \(E_5\) have degeneracy 2.

(c) In order to minimize the total energy, the five electrons should occupy the states \((1, 1), (2, 1), (1, 2), (3, 1), (1, 3)\) respectively. The highest energy level that is occupied is then given by \(E_4 = \frac{5\hbar^2}{m} \left( \frac{\pi}{a} \right)^2\).

(d) In general, degeneracies correspond to symmetries so once the infinite square well is no longer symmetric in \(x\) and \(y\) directions i.e when \(a \neq b\), then there will be no degeneracy. To see this more explicitly note that for \(a \neq b\) Eq. (7) implies that, for instance, \(\psi_{3,1}\) and \(\psi_{1,3}\) do not have the same energy: \(E_{13} \neq E_{31}\). Consequently, in part (b), all of the first 8 states will correspond to different energies for \(a \neq b\). For part (c) the highest energy will also change.

III. ANGULAR MOMENTUM STATES [6]

We consider an angular wavefunction expressed in terms of spherical harmonics (for instance, see p. 183 in the text):

\[ \psi(\theta, \phi) = \frac{1}{2} Y_{1,1} + i \frac{1}{\sqrt{2}} Y_{1,0} - \frac{1}{2} Y_{1,-1} \]

(a) In order to determine the probability of finding the value of \(2\hbar^2\) upon measuring the square of the total orbital angular momentum \(L^2\) it is useful to recall that

\[ L^2 Y_{l,m} = l(l+1)\hbar^2 Y_{l,m} \]

so that the following relationships are true

\[ L^2 Y_{1,0} = 2\hbar^2 Y_{1,0}, \quad L^2 Y_{1,1} = 2\hbar^2 Y_{1,1}, \quad L^2 Y_{1,-1} = 2\hbar^2 Y_{1,-1}. \]

(b) Arguing similarly as in part (b) we first use \(L_y Y_{l,m} = m\hbar Y_{l,m}\) in order to calculate

\[ L_y Y_{1,0} = 0, \quad L_y Y_{1,1} = \hbar Y_{1,1}, \quad L_y Y_{1,-1} = -\hbar Y_{1,-1} \]

so that we see that the probability of a measurement of \(L_y\) yielding \(-\hbar\) is the same as the probability as finding the object described by this wavefunction in the state with \(Y_{1,-1}\). Since the pre-factor of \(Y_{1,-1}\) in \(\psi(\theta, \phi)\) is \(1/2\) we conclude that this probability is 25%.
(c) Recalling Eqs. (6.45) and (6.46) in Townsend, we have

\[ Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta. \quad (10) \]

Applying \( L_z = -i\hbar \partial / \partial \phi \) on these two functions then gives

\[-i\hbar \frac{\partial}{\partial \phi} Y_{1,1} = -i\hbar \sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} = \hbar Y_{1,1} \]
\[-i\hbar \frac{\partial}{\partial \phi} Y_{1,0} = 0. \quad (11)\]

Consequently, we conclude that \( \lambda_1 = \hbar \) and \( \lambda_2 = 0 \), as is consistent with \( L_z Y_{l,m} = m \hbar Y_{l,m} \).

(d) Plots of \( |Y_{1,1}|^2 \) and \( |Y_{1,0}|^2 \) are shown below.

IV. HYDROGEN WAVEFUNCTIONS [8]

Suppose an electron is in the \( n = 2, \ell = 1, m_\ell = 1 \) state of the hydrogen atom.

(a) The plots are shown as below.

(b) The radial probability density is given by \( r^2 |R_{2,1}|^2 \) and according to Eq. (6.78) in Townsend,

\[ R_{2,1} = \frac{1}{\sqrt{3}} \left( \frac{Z}{2a_0} \right)^{3/2} Z r e^{-Zr/2a_0}. \quad (12) \]
Consequently, we can evaluate

$$r^2 |R_{2,1}|^2 = \frac{1}{3} \left( \frac{Z}{2a_0} \right)^3 \frac{Z^2 r^4}{a_0^3} e^{-Zr/a_0}. \tag{13}$$

To find $r_{\text{max}}$, we perform the standard extremization procedure: first we take the derivative with respect to $r$, then we find a value of $r$ at which that derivative vanishes. More precisely

$$\frac{d}{dr} \left\{ r^2 |R_{2,1}|^2 \right\} = \frac{1}{3} \left( \frac{Z}{2a_0} \right)^3 \left( 4 - \frac{Z}{a_0} \right) \frac{Z^2 r^3}{a_0^3} e^{-Zr/a_0} = 0 \Rightarrow r_{\text{max}} = \frac{4a_0}{Z}. \tag{14}$$

Note that for hydrogen, $Z = 1$.

(c) The energy of the photon is simply equal to the difference in energy between the $n = 1$ and $n = 2$ energy levels

$$E_{\text{ph}} = E_2 - E_1 = 13.6 \left( 1 - \frac{1}{4} \right) \text{eV} = 10.2 \text{ eV} \tag{15}$$

If the transition occurs between the energy eigenstates corresponding to $(n, l, m) = (2, 1, -1)$ and $(1, 0, 0)$ the energy of the photon will not change since, in the absence of external magnetic or electric fields, the energy eigenstates of the hydrogen atom only depend on the principal quantum number $n$ (recall that $E_n = -13.6/n^2$ eV). In other words, the relevant energy difference would still be $E_2 - E_1$.

(d) From low energy to high energy, the states of the 11 electrons of Na are described by

$$\begin{align*}
(1, 0, 0, \text{up}), (1, 0, 0, \text{down}), (2, 0, 0, \text{up}), (2, 0, 0, \text{down}), (2, 1, -1, \text{up}), (2, 1, -1, \text{down}), (2, 1, 0, \text{up}), (2, 1, 0, \text{down}), \\
(2, 1, 1, \text{up}), (2, 1, 1, \text{down}), (3, 0, 0, \text{up}) \text{ or } (3, 0, 0, \text{down}).
\end{align*} \tag{16}$$