Homework 7

PHYS 485: Fall 2017
Due Date: 11/29/2017

To recieve fully credit please write legibly and clearly, show all required steps and calculations and answer any explicit questions in full sentences.

I. HYDROGEN ATOM AND UNCERTAINTY [7]

In this problem you will use the uncertainty principle in order to *roughly* estimate the ground state energy (lowest energy level) and the radius of the hydrogen atom.

(a) Suppose that it is sufficient to model the energy levels of the hydrogen atom by simply taking

$$E(p, r) = \frac{p^2}{2m_\mathrm{e}} - \frac{e^2}{4\pi\epsilon_0 r} \tag{1}$$

where the first term is the kinetic energy of the hydrogen's single electron and the second is the Coulomb potential due to the positively charge nucleus. As a somewhat crude approximation, take the uncertainty principle to imply

$$pr = \hbar \tag{2}$$

and use this expression to write the energy in Eq. (1) as a function of p only. [1]

(b) Find p_min that minimizes $E(p)$ you determined above. [3]

(c) What de Broglie wavelength corresponds to p_min? [1]

(d) Use p_min to estimates the ground state energy of the hydrogen atom in eV by calculating $E(p_\mathrm{min})$. Similarly, use the given uncertainity relation, Eq. (2), to calculate r_{min}, the corresponding estimate for the hydrogen atom's radius. [2]

II. SPIN OPERATORS [7]

The matrices of spin operators can be expressed in the basis of S_z^op eigenstates $|\uparrow\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as follows:

$$S_x^\mathrm{op} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y^\mathrm{op} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z^\mathrm{op} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Additionally, we define the identity matrix as $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Show that the spin operator matrices obey the commutation relation

$$[S_z^\mathrm{op}, S_y^\mathrm{op}] = -i\hbar S_x^\mathrm{op}$$

where $[A, B] = AB - BA$ for any two operators A and B. [3]

(b) Calculate $(S_x^\mathrm{op})^2, (S_z^\mathrm{op})^2$, and $(S_y^\mathrm{op})^2$. How are these matrices related to each other and the identity matrix? [4]

III. SPIN EIGENVECTORS [7]

(a) Calculate the normalized eigenvectors of S_z^op using the matrix representation given in Problem 2. [2]

(b) Calculate the normalized eigenvectors of S_y^op using the matrix representation given in Problem 2. [2]

(c) A collection of silver atoms is passed through a Stern-Gerlach apparatus and filtered in such a way that the atoms are all in a state having eigenvalue $S_y = \hbar/2$. Now, if a measurement of the S_z component of spin is carried out on these atoms, what is the probability of obtaining $S_z = \hbar/2$? Explain. [3]

IV. SPIN AND LARMOR PRECESSION [14]

An electron starts off in an arbitrary normalized spin state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, expressed in the basis of S_z^{op} eigenstates,

$$\chi = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

in the presence of a magnetic field of strenght $\vec{B}$ aligned along the z-axis i.e. $\vec{B} = B\hat{z}$.

(a) Recall that the Hamiltonian for a spin-1/2 particle in a magnetic field reads

$$H = -\mu \cdot B = \frac{ge}{2m} B S_z^{\mathrm{op}}.$$

The states $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ are the eigenstates of H. What are their eigenenergies? [2]

(b) Consider the time-evolution of the state given

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2) e^{i\nu Bt/2} \\ \sin(\alpha/2) e^{-i\nu Bt/2} \end{pmatrix}$$

where α is defined by $a = \cos(\alpha/2),\ b = \sin(\alpha/2)$. The quantity νB is called the Larmor frequency. What is ν? [2]

(c) At some given time t, find the expectation values

$$\langle S_x^{\mathrm{op}} \rangle = \chi^\dagger S_x^{\mathrm{op}} \chi,\ \langle S_y^{\mathrm{op}} \rangle = \chi^\dagger S_y^{\mathrm{op}} \chi,\ \langle S_z^{\mathrm{op}} \rangle = \chi^\dagger S_z^{\mathrm{op}} \chi.$$

where $v^\dagger$ denotes the conjugate transpose of a vector:

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow v^\dagger = \begin{pmatrix} x^* & y^* \end{pmatrix}.$$

Express your answer in terms of α. [6] *Hint: Use the following identities to simplify your answers:*

$$\cos^2(x) - sin^2(x) = \cos(2x),$$
$$\cos(x)\sin(y) + \sin(x)\cos(y) = \sin(x+y).$$

(d) Consider these time-dependent expectation values of the components of $\vec{S}(t) = (S_x(t), S_y(t), S_z(t))$ and draw the $\vec{S}$ vector in Cartesian coordinates at some point in time. How does the position of this vector evolve as a function of time? Clearly indicate its motion in your drawing. [4]
This motion is called Larmor precession and it forms the basis of Magnetic Resonance Imaging (MRI).