Homework 8

PHYS 485: Fall 2017
Due Date: 12/8/2017

*To recieve fully credit please write legibly and clearly, show all required steps and calculations and answer any explicit questions in full sentences. Problems IV and V are **not mandatory** but you can get **extra homework credit** by attempting either or both.*

I. QUANTUM STATISTICS [18]

Two particles are distributed across four energy states: E_1, E_2, E_3, and E_4. In this problem you will learn how their (in)distinguishable nature and quantum statistics greatly dictate their distributions.
Consider three possible cases:

(i) Distinguishable particles: can be denoted by two different labels such as A and B, for example.

(ii) Bosons: can be denoted by the same letter or label. Keep in mind that there is *no restriction* on how many particles can occupy a single energy state. (For instance, two or three, or fifty, bosons can all occupy a single energy state E.)

(iii) Fermions: can be denoted them by the same letter. Keep in mind that *only one* particle can occupy a single energy state. (For instance, to distribute three fermions, three or more energy states are needed.)

For each of these cases

(a) Write out all possible configurations of particles. You can, for example, denote each energy state as a box and the particle as a letter to put inside the box (or not). Refer to Fig. 7.9 in Townsend for an example of such representation. [6]

(b) A microstate represents a particular configuration of the two particles occupying specific energy states. How many microstates are there for each case? [6]

(c) For each case, what is the probability of finding two particles in the same energy state? (The probability is the number of microstates that respect this configuration divided by the total number of microstates.) Based on these probabilities, comment on the differences between distinguishable particles, bosons, and fermions. [6]

Instead of two particles and four energy states, if you were to scale up this analysis to many particles, many energy states and total fixed energy, you would obtain the different kinds of distribution functions for the three cases (Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac) discussed in Eqs. (7.43-5) in Townsend. This system could act as a thermal reservoir when in contact with a smaller system.

II. BOSE EINSTEIN CONDENSATION [8]

Unlike fermions which are a more "solitary" particle governed by the Pauli exclusion principle, bosons are likely to "bunch up" and macroscopically occupy the same energy state at low temperatures. When a large number of bosons does so i.e. when most of them are occupying the lowest available energy state, we do not speak of a gas of bosons anymore but rather a new, fundamentally quantum, state of matter called the Bose Einstein condensate. Such a state of matter can only be achieved at low temperatures – temperatures below some critical temperature T_c. In this problem, you will estimate T_c without having to resort to complicated statistical calculations.

(a) First, we will use the fact that in three dimensions the "equipartition theorem" states that the average kinetic energy of the bosons at some temperature T is given by

$$\frac{1}{2}mv^2 = \frac{3}{2}kT.$$

Use this to express the de Broglie wavelength $\lambda_{\mathrm{dB}} = h/p$ in terms of T, m and fundamental constants. [2]

(b) At the critical temperature for Bose-Einstein condensation, the spacing between atoms is comprable to their de Broglie wavelength – their quantum wave-like nature cannot be ignored! What is the spacing between atoms d for a cloud of N bosons confined in a volume V? [2]

(c) By equating the spacing between atoms d and their de Broglie wavelength λ_{dB} find T_c as a function of N, V, m and fundamental constants. [1]

(d) Calculate T_c for a cloud of $N = 2 \times 10^7$ atoms of ${}^4\text{He}$ in a volume of $10 \ \mu\text{m}^3$. How many orders of magnitude (powers of 10) is this temperature lower than room temperature $T_{\text{room}} \approx 300$ K? [3]

In laboratory settings, such temperatures (and thousands of times lower) are achieved by means of manipulating bosons with laser light and magnets – some of the labs on the third floor of Loomis are among the coldest places in the world!

III. FERMIONS AND THEIR PROPERTIES [14]

(a) Consider the Fermi-Dirac distribution function given by

$$n(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1}.$$

What is its physical meaning? Plot the distribution function as a function of energy at

   (i) zero temperature
   (ii) $E_F \gg k_B T$
   (iii) $E_F \ll k_B T$.

For (ii), show the width $k_B T$ on your plot on both sides of the Fermi energy, E_F. This region should be where (ii) deviates from (i). [6]

(b) Consider a free-electron model describing conduction electrons in a metal. The density-of-states (number of states within a "sliver" of energy centered around E) is given by

$$D(E)dE = \frac{V(2m)^{3/2}}{2\hbar^3 \pi^2} \sqrt{E} dE.$$

At zero temperature, the total number of particles is given by $N = \int_0^{E_F} D(E)dE$. Find a relationship between the Fermi energy and density of electrons, N/V. [2]

(c) For silver, having density 10.49 g/cm³, find the Fermi energy. At what temperature does $k_B T$ become comparable to this energy? This is the temperature at which the electronic properties of the metal would begin to disintegrate. [3]

(d) Now find the net energy of the conduction electrons at zero temperature and express it in terms of the Fermi energy and number of particles [3]

$$E_{\text{net}} = \int_0^{E_F} E D(E)dE.$$

IV. *EXTRA CREDIT*: BOLTZMANN STATISTICS [8]

In this problem we consider a molecule modeled as a system having three equally spaced energy levels $E_1 = \epsilon, E_2 = 2\epsilon$ and $E_3 = 3\epsilon$ (corresponding to, for instance, some rotational degrees of freedom) that is in contact with a large thermal reservoir of temperature T. In such a situation, we can calculate the probability of the molecule being in any of these energy levels E_i by using the Boltzmann factor $e^{-E_i/kT}$ as follows:

$$P_i = \frac{d_i e^{-E_i/kT}}{\sum_j d_j e^{-E_j/kT}}$$

where d_j denotes the degeneracy of a state with energy E_j. Note that, in this problem, E_1, E_2 and E_3 all have degeneracy 1 i.e. $d_1 = d_2 = d_3 = 1$.

(a) What is the probability P_1 that the molecule will be in its lowest energy state E_1? What is the probability P_3 that the molecule will be in its highest energy state E_3? [4]

(b) How would your answers change if this molecule did have some degenerate energy states? More precisely, what is the value of P_1 if we consider a different molecule having available energy states $E_1 = \epsilon, E_1' = \epsilon, E_2 = 2\epsilon$ and $E_3 = 3\epsilon$? *Hint: Now $d_1 = 2$* [2]

(c) What is P_1 when $T \to \infty$? How do P_2 and P_3 compare to P_1 for $T \to \infty$? [2]

V.   *EXTRA CREDIT*: SEMICONDUCTORS [8]

A semiconductor is characterized by an energy gap E_g at the Fermi energy. At zero temperature, states just below the gap (valence band) are filled and states above the gap are empty (conduction band). At finite temperature, due to thermal excitations, the average number of electrons per state excited into the conduction band is given by

$$n(E_g) = \frac{1}{e^{E_g/2k_B T} + 1}.$$

For $E_g \gg k_B T$, we have $n(E_g) \simeq e^{-E_g/2k_B T}$.

(a) Assuming that $E_g \gg k_B T$, calculate the average number of electrons per state at the bottom of the conduction band for geranium, having $E_g = 0.66$ eV at room temperature i.e. $T_{\mathrm{rm}} = 300$ K. [2]

(b) Repeat the same calculation for $T = 360$ K. [2]

(c) What is the percentage change in absolute temperature between T_{rm} and T? [2]

(d) What percent change in value of $n(E_g)$ does this change in absolute temperature produce? Discuss. [2]