high intensities, so that the granular quantum phenomena contradict the continuum classical view more dramatically there.

In analogy to Einstein's view of radiation, Max Born proposed a similar uniting of the wave-particle duality for matter. This came several years after Schroedinger developed his generalization of de Broglie's postulate, called quantum mechanics. We shall examine Schroedinger's theory quantitatively in later chapters. Here we wish merely to use Born's idea in a qualitative way to set the stage conceptually for the subsequent detailed analysis.

Let us associate more than just a wavelength and frequency with matter waves. We do this by introducing a function representing the de Broglie wave, called the *wave function* Ψ. For particles moving in the x direction with a precise value of linear momentum and energy, for example, the wave function can be written as a simple sinusoidal function of amplitude A, such as

$$\Psi(x,t) = A \sin 2\pi\left(\frac{x}{\lambda} - \nu t\right) \tag{3-4a}$$

This is analogous to

$$\mathscr{E}(x,t) = A \sin 2\pi\left(\frac{x}{\lambda} - \nu t\right) \tag{3-4b}$$

for the electric field of a sinusoidal electromagnetic wave of wavelength λ, and frequency ν, moving in the positive x direction. The quantity $\overline{\Psi^2}$ will play a role for matter waves analogous to that played by $\overline{\mathscr{E}^2}$ for waves of radiation. That quantity, the average of the square of the wave function of matter waves, is a measure of the probability of finding a particle in unit volume at a given place and time. Just as $\mathscr{E}$ is a function of space and time, so is Ψ; and, as we shall see later, just as $\mathscr{E}$ satisfies a wave equation, so does Ψ (Schroedinger's equation). The quantity $\mathscr{E}$ is a (radiation) wave associated with a photon, and Ψ is a (matter) wave associated with a material particle.

As Born says: "According to this view, the whole course of events is determined by the laws of probability; to a state in space there corresponds a definite probability, which is given by the de Broglie wave associated with the state. A mechanical process is therefore accompanied by a wave process, the guiding wave, described by Schroedinger's equation, the significance of which is that it gives the probability of a definite course of the mechanical process. If, for example, the amplitude of the guiding wave is zero at a certain point in space, this means that the probability of finding the electron at this point is vanishingly small."

Just as in the Einstein view of radiation we do not specify the exact location of a photon at a given time, but specify instead by $\overline{\mathscr{E}^2}$ the probability of finding a photon at a certain location at a given time, so here in Born's view we do not specify the exact location of a particle at a given time, but specify instead by $\overline{\Psi^2}$ the probability of finding a particle at a certain location at a given time. Just as we are accustomed to adding wave functions ($\mathscr{E}_1 + \mathscr{E}_2 = \mathscr{E}$) for two superposed electromagnetic waves whose resultant intensity is given by $\mathscr{E}^2$, so we shall add wave functions for two superposed matter waves ($\Psi_1 + \Psi_2 = \Psi$) whose resultant intensity is given by Ψ^2. That is, a *principle of superposition* applies to matter as well as to radiation. This is in accordance with the striking experimental fact that matter exhibits interference and diffraction properties, a fact that simply cannot be understood on the basis of ideas in classical mechanics. Because waves can be superposed either constructively (in phase) or destructively (out of phase), two waves can combine either to yield a resultant wave of large intensity or to cancel, but two classical particles of matter cannot combine in such a way as to cancel.

The student might accept the logic of this fusion of wave and particle concepts but nevertheless ask whether a probabilistic or statistical interpretation is necessary.

It was Heisenberg and Bohr who, in 1927, first showed how essential the concept of probability is to the union of wave and particle descriptions of matter and radiation. We investigate these matters in succeeding sections.

3-3 THE UNCERTAINTY PRINCIPLE

The use of probability considerations is not foreign to classical physics. Classical statistical mechanics makes use of probability theory, for example. However, in classical physics the basic laws (such as Newton's laws) are deterministic, and statistical analysis is simply a practical device for treating very complicated systems. According to Heisenberg and Bohr, however, the probabilistic view is the fundamental one in quantum physics and determinism must be discarded. Let us see how this conclusion is reached.

In classical mechanics the equations of motion of a system with given forces can be solved to give us the position and momentum of a particle at all values of the time. All we need to know are the precise position and momentum of the particle at some value of the time $t = 0$ (the initial conditions) and the future motion is determined exactly. This mechanics has been used with great success in the macroscopic world, for example in astronomy, to predict the subsequent motions of objects in terms of their initial motions. Note, however, that in the process of making observations the observer interacts with the system. An example from contemporary astronomy is the precise measurement of the position of the moon by bouncing radar from it. The motion of the moon is disturbed by the measurement, but due to the very large mass of the moon the disturbance can be ignored. On a somewhat smaller scale, as in a very well-designed macroscopic experiment on earth, such disturbances are also usually small, or at least controllable, and they can be taken into account accurately ahead of time by suitable calculations. Hence, it was naturally assumed by classical physicists that in the realm of microscopic systems the position and momentum of an object, such as a electron, could be determined precisely by observations in a similar way. Heisenberg and Bohr questioned this assumption.

The situation is somewhat similar to that existing at the birth of relativity theory. Physicists spoke of length intervals and time intervals, i.e., space and time, without asking critically how one actually measures them. For example, they spoke of the simultaneity of two separated events without even asking how one would physically go about establishing simultaneity. In fact, Einstein showed that simultaneity was not an absolute concept at all, as had been assumed previously, but that two separated events that are simultaneous to one observer occur at different times to another observer moving with respect to the first. Simultaneity is a relative concept. Similarly then, we must ask ourselves how we actually measure position and momentum.

Can we determine by actual experiment at the same instant both the position and momentum of matter or radiation? The answer given by quantum theory is: not more accurately than is allowed by the Heisenberg *uncertainty principle*. There are two parts to this principle, also called the indeterminacy principle. The first has to do with the simultaneous measurement of position and momentum. It states that experiment cannot simultaneously determine the exact value of a component of momentum, p_x say, of a particle and also the exact value of its corresponding coordinate, x. Instead, our precision of measurement is inherently limited by the measurement process itself such that

$$\Delta p_x \Delta x \geq \hbar/2 \tag{3-5}$$

where the momentum p_x is known to within an uncertainty of Δp_x and the position x at the same time to within an uncertainty Δx. Here $\hbar$ (read h-bar) is a shorthand symbol for $h/2\pi$, where h is Planck's constant. That is

$$\hbar \equiv h/2\pi$$

There are corresponding relations for other components of momentum, namely $\Delta p_y \Delta y \geq \hbar/2$ and $\Delta p_z \Delta z \geq \hbar/2$, and for angular momentum as well. It is important to realize that this principle has nothing to do with improvements in instrumentation leading to better simultaneous determinations of p_x and x. Rather the principle says that even with ideal instruments we can never in principle do better than $\Delta p_x \Delta x \geq \hbar/2$. Note also that the *product* of uncertainties is involved, so that, for example, the more we modify an experiment to improve our measure of p_x, the more we give up ability to determine x accurately. If p_x is known exactly we know nothing at all about x (i.e., if $\Delta p_x = 0$, $\Delta x = \infty$). Hence, *the restriction is not on the accuracy to which x or p_x can be measured, but on the product $\Delta p_x \Delta x$ in a simultaneous measurement of both.*

The second part of the uncertainty principle has to do with the measurement of the energy E and the time t required for the measurements, as for example, the time interval Δt during which a photon of energy spread ΔE is emitted from an atom. In this case

$$\Delta E \Delta t \geq \hbar/2 \tag{3-6}$$

where ΔE is the uncertainty in our knowledge of the energy E of a system and Δt the time interval characteristic of the rate of change in the system.

Heisenberg's relations will be shown later to follow from the de Broglie postulate plus simple properties common to all waves. Because the de Broglie postulate is verified by the experiments we have already discussed, it is fair to say that the uncertainty principle is grounded in experiment. We shall also consider soon the consistency of the principle with other experiments. Notice first, however, that it is Planck's constant h that again distinguishes the quantum results from the classical ones. If h, or $\hbar$, in (3-5) and (3-6) were zero, there would be no basic limitation on our measurement at all, which is the classical view. Again it is the smallness of h that takes the principle out of the range of our ordinary experiences. This is analogous to the smallness of the ratio v/c in macroscopic situations taking relativity out of the range of ordinary experience. In principle, therefore, classical physics is of limited validity and in the microscopic domain it will lead to contradictions with experimental results. For if we cannot determine x and p simultaneously, then we cannot specify the initial conditions of motion exactly; therefore, we cannot precisely determine the future behavior of a system. Instead of making deterministic predictions, we can only state the possible results of an observation, giving the relative probabilities of their occurrence. Indeed, since the act of observing a system disturbs it in a manner that is not completely predictable, the observation changes the previous motion of the system to a new state of motion which cannot be completely known.

Let us now illustrate the physical origin of the uncertainty principle. With the insight thereby gained we shall better appreciate a more formal proof given in the following section. First, we use a thought experiment due to Bohr to verify (3-5). Let us say that we wish to measure as accurately as possible the position of a "point" particle, like an electron. For greatest precision we use a microscope to view the electron, as in Figure 3-6. To see the electron we must illuminate it, for it is actually the light photon scattered by the electron that the observer sees. At this stage, even before any calculations are made, we can see the uncertainty principle emerge. The very act of observing the electron disturbs it. The moment we illuminate the electron, it recoils because of the Compton effect, in a way that we shall soon find cannot be completely determined. If we don't illuminate the electron, however, we don't see (detect) it. Hence the uncertainty principle refers to the measuring process itself, and it expresses the fact that there is always an undetermined interaction between observer and observed; there is nothing we can do to avoid the interaction or to allow for it ahead of time. In the case at hand we can try to reduce the disturbance to the electron as much as possible by using a very weak source of light. The very weakest we can get

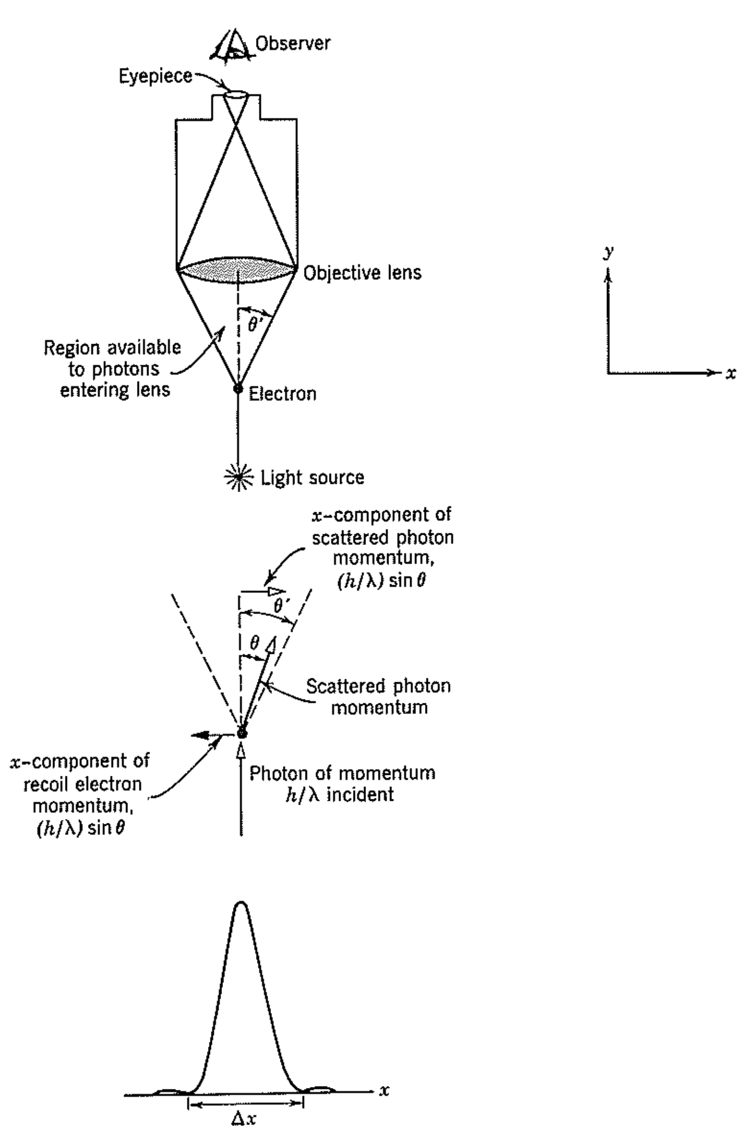

Figure 3-6 Bohr's microscope thought experiment. *Top:* The apparatus. *Middle:* The scattering of an illuminating photon by the electron. *Bottom:* The diffraction pattern image of the electron seen by the observer.

is to assume that we can see the electron if only *one* scattered photon enters the objective lens of the microscope. The magnitude of the momentum of the photon is $p = h/\lambda$. But the photon may have been scattered *anywhere* within the angular range $2\theta'$ subtended by the objective lens at the electron. This is why the interaction cannot be allowed for. Hence, we find that the x component of the momentum of the photon can vary from $+p \sin \theta'$ to $-p \sin \theta'$ and is uncertain after the scattering by an amount

$$\Delta p_x = 2p \sin \theta' = (2h/\lambda) \sin \theta'$$

Conservation of momentum then requires that the electron receive a recoil momentum in the x direction that is equal in magnitude to the x momentum change in the photon and, therefore, the x momentum of the electron is uncertain by this same amount. Notice that to reduce Δp_x we can use light of longer wavelength, or use a microscope with an objective lens subtending a smaller angle.

What about the location along x of the electron? Recall that a microscope's image of a point object is not a point, but a diffraction pattern; the image of the electron is "fuzzy." The resolving power of a microscope determines the ultimate accuracy to which the electron can be located. If we take the width of the central diffraction maximum as a measure of the uncertainty in x, a well-known expression for the resolving power of a microscope gives

$$\Delta x = \lambda/\sin\theta'$$

(Note that, since $\sin\theta \simeq \theta$, this is an example of the general relation $a \simeq \lambda/\theta$ between the characteristic dimension in a diffraction apparatus, the wavelength of the diffracted waves, and the diffraction angle.) The one scattered photon at our disposal must have originated then *somewhere* within this range from the axis of the microscope, so the uncertainty in the electron's location is Δx. (We cannot be sure exactly where any one photon originates even though in a large number of repetitions of the experiment the photons forming the total image will produce the diffraction pattern shown in the figure.) Notice that to reduce Δx we can use light of shorter wavelength, or a microscope with an objective lens subtending a larger angle.

If now we take the product of the uncertainties we find

$$\Delta p_x \Delta x = \left(\frac{2h}{\lambda}\sin\theta'\right)\left(\frac{\lambda}{\sin\theta'}\right) = 2h \tag{3-7}$$

in reasonable agreement with the ultimate limit $\hbar/2$ set by the uncertainty principle. We cannot *simultaneously* make Δp_x and Δx as small as we wish, for the procedure that makes one small makes the other large. For instance, if we use light of short wavelength (e.g., γ rays) to reduce Δx by obtaining better resolution, we increase the Compton recoil and increase Δp_x, and conversely. Indeed, the wavelength λ and the angle θ' subtended by the objective lens do not even appear in the result. In practice an experiment might do much worse than (3-7) suggests, for that result represents the very ideal possible. We arrive at it, however, from genuinely measurable physical phenomena, namely the Compton effect and the resolving power of a lens.

There really should be no mystery in the student's mind about our result. It is a direct result of quantization of radiation. We had to have at least one photon illuminating the electron, or else no illumination at all; and even one photon carries a momentum of magnitude $p = h/\lambda$. It is this single scattered photon that provides the necessary interaction between the microscope and the electron. This interaction disturbs the particle in a way that cannot be exactly predicted or controlled. As a result, the coordinates and momentum of the particle cannot be completely known after the measurement. If classical physics were valid, then since radiation is regarded there as continuous rather than granular, we could reduce the illumination to arbitrarily small levels and deliver arbitrarily small momentum while using arbitrarily small wavelengths for "perfect" resolution. In principle there would be no simultaneous lower limit to resolution or momentum recoil and there would be no uncertainty principle. But we cannot do this; the single photon is indivisible. Again we see, from $\Delta p_x \Delta x \geq \hbar/2$, that Planck's constant is a measure of the minimum uncontrollable disturbance that distinguishes quantum physics from classical physics.

Now let us consider (3-6) relating energy and time uncertainties. For the case of a free particle we can obtain (3-6) from (3-5), which relates position and momentum, as follows. Consider an electron moving along the x axis whose energy we can write as $E = p_x^2/2m$. If p_x is uncertain by Δp_x, then the uncertainty in E is given by $\Delta E = (p_x/m)\Delta p_x = v_x \Delta p_x$. Here v_x can be interpreted as the recoil velocity along x of the electron which is illuminated with light in a position measurement. If the time interval required for the measurement is Δt, then the uncertainty in its x position is $\Delta x = v_x \Delta t$. Combining $\Delta t = \Delta x/v_x$ and $\Delta E = v_x \Delta p_x$, we obtain $\Delta E \Delta t = \Delta p_x \Delta x$. But $\Delta p_x \Delta x \geq \hbar/2$. Hence

$$\Delta E \Delta t \geq \hbar/2$$

Example 3-3. The speed of a bullet ($m = 50$ g) and the speed of an electron ($m = 9.1 \times 10^{-28}$ g) are measured to be the same, namely 300 m/sec, with an uncertainty of 0.01%. With what fundamental accuracy could we have located the position of each, if the position is measured simultaneously with the speed in the same experiment?

▶For the electron

$$p = mv = 9.1 \times 10^{-31} \text{ kg} \times 300 \text{ m/sec} = 2.7 \times 10^{-28} \text{ kg-m/sec}$$

and

$$\Delta p = m\Delta v = 0.0001 \times 2.7 \times 10^{-28} \text{ kg-m/sec} = 2.7 \times 10^{-32} \text{ kg-m/sec}$$

so that

$$\Delta x \geq \frac{h}{4\pi\Delta p} = \frac{6.6 \times 10^{-34} \text{ joule-sec}}{4\pi \times 2.7 \times 10^{-32} \text{ kg-m/sec}} = 2 \times 10^{-3} \text{ m} = 0.2 \text{ cm}$$

For the bullet

$$p = mv = 0.05 \text{ kg} \times 300 \text{ m/sec} = 15 \text{ kg-m/sec}$$

and

$$\Delta p = 0.0001 \times 15 \text{ kg-m/sec} = 1.5 \times 10^{-3} \text{ kg-m/sec}$$

so that

$$\Delta x \geq \frac{h}{4\pi\Delta p} = \frac{6.6 \times 10^{-34} \text{ joule-sec}}{4\pi \times 1.5 \times 10^{-3} \text{ kg-m/sec}} = 3 \times 10^{-32} \text{ m}$$

Hence, for macroscopic objects such as bullets the uncertainty principle sets no practical limit to our measuring procedure, Δx in this example being about 10^{-17} times the diameter of a nucleus; but, for microscopic objects such as electrons, there are practical limits, Δx in this example being about 10^7 times the diameter of an atom. ◀

3-4 PROPERTIES OF MATTER WAVES

In this section we shall derive the uncertainty principle relations by combining the de Broglie-Einstein relations, $p = h/\lambda$ and $E = h\nu$, with simple mathematical properties that are universal to all waves. We begin a development of these properties by calling attention to an apparent paradox.

The velocity of propagation w of a wave with wavelength and frequency λ and ν is given by the familiar relation, which we shall verify later

$$w = \lambda\nu \tag{3-8}$$

Let us evaluate w for a de Broglie wave associated with a particle of momentum p and total energy E. We obtain

$$w = \lambda\nu = \frac{h}{p}\frac{E}{h} = \frac{E}{p}$$

Now assume the particle is moving at nonrelativistic velocity v in a region of zero potential energy. (The validity of our conclusions will not be limited by these assumptions.) Evaluating p and E in terms of v and the mass m of the particle, we find

$$w = \frac{E}{p} = \frac{mv^2/2}{mv} = \frac{v}{2} \tag{3-9}$$

This result seems disturbing because it appears that the matter wave would not be able to keep up with the particle whose motion it controls. However, there is really no difficulty, as the following argument shows.

Imagine that a particle is moving along the x axis under the influence of no force because its potential energy has the constant value zero. Moving along that axis is also its associated matter wave. Assume, for the sake of this thought experiment, that we have distributed along the axis a set of (hypothetical) instruments which are capable of measuring the amplitude of the matter wave. At some time, say $t = 0$, we record