Midterm Exam 2 Formula Sheet

April. 4, 2023

Reference formulae

Time-dependent Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi(x,t)$
Normalization: $\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 = 1$
Expectation values: $\langle\mathcal{O}\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x,t)\mathcal{O}\psi(x,t)$
standard deviation σ: $\sigma_{\mathcal{O}}^2 = \langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2$
Time-independent Schrödinger equation: $H\psi_n(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n(x) + V(x)\psi_n(x) = E_n\psi_n(x)$
$\psi(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}$
Orthonormality: $\int_{-\infty}^{\infty} dx\, \psi_m^*(x)\psi_n(x) = \delta_{mn}$

Operators: momentum $p \leftrightarrow -i\hbar\frac{\partial}{\partial x}$; position $x \leftrightarrow x$; Hamiltonian $H \leftrightarrow p^2/2m + V(x,t)$
Commutator $[A,B] = AB - BA$; $[x,p] = i\hbar$
Uncertainty $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2$
$\frac{d}{dt}\langle Q\rangle = (i/\hbar)\langle[H,Q]\rangle + \langle\partial Q/\partial t\rangle$
Infinite square well, $V(x) = 0$ for $0 < x < L$, $V(x) = \infty$ elsewhere:
$\psi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$, $E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$

Free particle, $V = 0$. Momentum eigenstates $\psi_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$
Continuum orthonormality, $\int_{-\infty}^{\infty} dx\, \psi_{k_1}^*(x)\psi_{k_2}(x) = \delta(k_1 - k_2)$

Integrals: Gaussian, $\int_{-\infty}^{\infty} dx\, e^{-(\alpha x^2+\beta x)} = \sqrt{\pi/\alpha}\, e^{\beta^2/4\alpha}$ for $\mathrm{Re}\,\alpha > 0$
$\int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} = \sqrt{\pi/4\alpha^3}$
Delta function, $\delta(x) = 0$ for $x \neq 0$, ∞ for $x = 0$, $\int_{-\infty}^{\infty} dx\, \delta(x-a)f(x) = f(a)$
Fourier transform: $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\tilde{f}(k)e^{ikx}$, $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(x)e^{-ikx}$

Simple harmonic oscillator, $V(x) = \frac{1}{2}m\omega^2x^2$: define $x_0^2 \equiv \hbar/m\omega$, then
$\psi_n(x) = A_ne^{-x^2/2x_0^2}H_n(x/x_0)$, $A_n = (2^nn!x_0\sqrt{\pi})^{-1/2}$, Hermite polynomials $H_n(y)$: $H_0(y) = 1$
$H_1(y) = 2y$, $H_2(y) = 4y^2 - 2$, $H_3(y) = 8y^3 - 12y$, $H_4(y) = 16y^4 - 48y^2 + 12$
$E_n = (n + 1/2)\hbar\omega$
Raising and lowering operators,

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{x_0p}{\hbar}\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{x_0p}{\hbar}\right), \qquad [a,a^\dagger] = 1$$

Laplacian in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Angular momentum operators
Commutation relations: $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$, $[L_z, L^2] = 0$
In spherical coordinates:

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Raising and lowering operators,

$$L_\pm = L_x \pm iL_y = \hbar e^{\pm i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right), \qquad [L_z, L_\pm] = \pm\hbar L_\pm$$

First few spherical harmonics:

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \qquad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi} \qquad Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

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Midterm 2

Problem 1 - Finite dimensional Hilbert space [25 pts]

Consider a Hilbert space consisting of three orthonormal states $\{|1\rangle, |2\rangle, |3\rangle\}$

(a) A linear operator is defined on this Hilbert space:

$$Z|1\rangle = -|2\rangle + |3\rangle \qquad Z|2\rangle = -|1\rangle + |3\rangle \qquad Z|3\rangle = |3\rangle \tag{1}$$

write a matrix representation of this operator.

(b) Find $Z^\dagger$ by giving expressions for $Z^\dagger|1\rangle, Z^\dagger|2\rangle, Z^\dagger|3\rangle$.

(c) The system is prepared in the state $\frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - i|3\rangle)$. An unspecified observable is measured and found to be some definite [unspecified] value. Directly after the measurement the system has collapsed to the state $\frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$. What was the probability of this measurement outcome?

Problem 2 - Simple Harmonic Oscillator [25 pts]

A simple harmonic oscillator is prepared in the initial state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{2}$$

where $|n\rangle$ are eigenstates of the number operator $\hat{N}$.

(a) Compute the average "occupation" $= \langle\psi|\hat{N}|\psi\rangle$. What is the average energy of the SHO?

(b) Use creation and annihilation operators to compute $\langle\psi|\hat{x}|\psi\rangle$

(c) [This problem requires little to no algebra] Consider the state $\propto a^\dagger a a^\dagger a^\dagger a a^\dagger|0\rangle$. The proportionality constant is unspecified but needed to make this state normalized. A measurement of the energy is made on this state, what are the possible outcomes? What can you say about the state $\propto a^\dagger a a a^\dagger a a^\dagger|0\rangle$?

Problem 3 - Angular momentum [25 pts]

Consider a particle on a sphere specified by the angles (θ, ϕ) used in spherical coordinates. The particle is in some state $|\psi\rangle$

(a) The angular momentum L_z, L^2 are simultaneously measured and the state collapses to $|\ell, m\rangle$ for integer ℓ, m. Write an algebraic expression for the probability of this outcome - firstly use the abstract Hilbert space langauge and secondly use the position basis representation where $\langle\theta, \phi|\psi\rangle = \psi(\theta, \phi)$.

(b) The value of L^2 is measured to be $12\hbar^2$. What are the possible values of L_z?

(c) Compute the commutator of $\frac{1}{\sqrt{2}}(L_x + L_y)$ and $L_x^2 + L_y^2$.