Fourier Series as an Inner Product Space

- **Space** : $|f\rangle \equiv$ **L-periodic functions** $f(x)$ that are periodic over $x = \left[-\frac{L}{2} \rightarrow \frac{L}{2}\right]$

- $\mathbb{R}$ **Real Basis** : $|n\rangle \equiv \begin{cases} \sin\left(\frac{n 2\pi x}{L}\right) & n = 1,\ldots,\infty \\ 1/\sqrt{2} & n = 0 \\ \cos\left(\frac{n 2\pi x}{L}\right) & n = -1,\ldots,-\infty \end{cases}$

- $\mathbb{R}$ **Inner Product** : $\langle g|f\rangle \equiv \frac{2}{L}\int_{-L/2}^{L/2} g(x) f(x) dx$

- $\mathbb{C}$ **Complex Basis** : $|n\rangle = \exp\left(i\,\frac{n 2\pi x}{L}\right)$, n = -∞...+∞

- $\mathbb{C}$ **Inner Product** : $\langle g|f\rangle \equiv \frac{1}{L}\int_{-L/2}^{L/2} g^*(x) f(x) dx$

➔ **Basis is Orthonormal** : $\langle n|m\rangle = \delta_{nm}$

➔ **Completeness** : any $|f\rangle = \sum_{n=-\infty}^{+\infty} |n\rangle\langle n|f\rangle$

Fourier Transform as an Inner Product Space

* take limit $L \rightarrow \infty$ of above with $k \equiv 2\pi n / L$

- **Space** : $|f\rangle \equiv$ **real or complex functions** $f(x)$ [1]

- **Basis #1** : $|k\rangle = \exp(ikx)$ for $k = -\infty$ to $+\infty$

- **Inner Product #1** : $\langle g|f\rangle \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(x) f(x) dx$

- **Basis #2** : $|k\rangle = \frac{1}{\sqrt{2\pi}}\exp(ikx)$ for $k = -\infty$ to $+\infty$

- **Inner Product #2** : $\langle g|f\rangle \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g^*(x) f(x) dx$

➔ **Basis is Orthonormal** : $\langle k_1|k_2\rangle = \delta_{k_1 k_2}$

➔ **Completeness** : any $|f\rangle = \int_{-\infty}^{+\infty} dk\, |k\rangle\langle k|f\rangle$

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[1] The fine print: $f(x)$ must be piecewise-continuous, differentiable, and absolutely integrable (i.e. $\int |f(x)|\, dx$ must be finite)