Physics 487 – Homework #2 **due in course homework box by Fri 1 pm**

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: our master rule for homework and exams is **NO WORK = NO POINTS**. *However you may always use any relation on the 3D-calculus and 1D-math formula sheets without proof; both are posted in the same place you found this homework. Finally please write your* **NAME** *and* **DISCUSSION SECTION** *on your solutions.* ☺

Just like last week, you may use anything from the **486 final formula sheets**, but try not to … see how far you can get on your Desert Island first. ☺

Also, *unless otherwise specified* you may use wolframalpha.com or similar tool to evaluate your integrals **after you set them up** in a form that can be **directly entered** into such tools. (Just remember that no electronic tools will be available on exams.)

Problem 1 : Spherical Warmup

3D physical systems often have **spherical symmetry**[1]. In such cases, (r,θ,ϕ) is the perfect coordinate system to use. Also, have a look at the file Formulae-UIUC-QualExam.pdf on our website. There's not a lot there, but you *will* find all the standard differential operators (grad, div, curl, Laplacian) in spherical and cylindrical coordinates, and you will also find the **spherical harmonic** functions $Y_l^m(\theta,\phi)$. That's nice. ☺ We can build a little "emergency room" on our desert island that has the four pages of Formulae-UIUC-QualExam.pdf on the wall as life-saving devices. ☺

(a) By definition, the spherical harmonic $Y_l^m(\theta,\phi)$ is the simultaneous **eigenfunction** of the **operator L^2** with **eigenvalue _____** and the **operator L_z** with **eigenvalue _____**. Do whatever reading or meditating you need to to fill in those two eigenvalue blanks (no explanation necessary), ponder some more to obtain the operators L^2 and L_z in spherical coordinates (still no explanation necessary), then show explicitly that your answers are correct for the spherical harmonic $Y_2^{-2}(\theta,\phi)$ (which you can get from the 486 or UIUC-Qual formula sheets).

(b) The spherical harmonics have strange numerical factors in front of them. Those factors are needed to **normalize** the functions. What "normalization" do they obey? Show explicitly that $Y_2^{-2}(\theta,\phi)$ is, in fact, normalized to 1 by performing the appropriate integral. (You figure out what that appropriate integral is … perhaps by pondering the simplest spherical harmonic Y_0^0 and meditating on where you've seen "4π" before.) NOTE: You may *not* use an electronic device to do the integral in question as you really don't need to.

Problem 2: Spherical-Shell Qual Problem

people at Labor Day office hours: this was problem 4

A particle of mass m is trapped in a spherical shell by impenetrable potential walls at the radii $r = a$ and $r = R$:

$$V(r) = \begin{cases} 0 & a < r < R \\ \infty & \text{elsewhere} \end{cases}$$

Calculate the energy eigenfunctions $\psi_n(r, \theta, \phi)$ and energy eigenvalues E_n that the particle may occupy when it has zero angular momentum (i.e. when it is in an eigenstate of L^2 with eigenvalue 0).

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[1] Jargon-busting: A **spherically symmetric** system is one that is unchanged by any rotation around any axis that passes through some particular point.

Problem 3 : 1D-Coulomb Qual Problem *this is the 2nd question on Discussion 2, with additional parts*

An electron moves in one dimension (x) and is confined to the right half-space ($x > 0$) where it has potential energy $V(x) = -e^2 / 4x$, where e is the charge on an electron in appropriate units. "This is the potential energy of an electron outside a perfect conductor", says the original qual problem ... which is exactly the sort of fascinating tidbit of information that you want to read all about in the middle of an exam. ;-)

(a) Find the ground state energy. HINT: Is the ground state of a particle in an *attractive* potential likely to be a *bound* state or a *free* state? Hmmm … If you have no idea what I'm hinting at, ask in class or in office hours!!

(b) Find the expectation value of position, $< x >$, in the ground state.

(c) Find the probability density, $\rho(p_x)$, where p_x is the x-component of momentum.

NOTE: You don't have to keep writing "p_x"; for brevity, go ahead and use "p" while you do all the algebra ... just remember that it refers to a momentum component (which can be positive or negative) not a magnitude (which is never negative).

HINTs: Normalization. Dirac normalization.

(d) Calculate $\langle p_x \rangle$ and $\langle p^2 \rangle$.

HINT: one needs an integral, the other needs only a sketch. Or a physical argument. Never integrate when a sketch or a physical argument is sufficient. ☺

Problem 4 : Shallow-Well Qual Problem

A particle of mass m is constrained to move in one dimension (x) and experiences this potential energy:

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| > a \end{cases}$$ where a and V_0 are known positive constants.

This is a square well that is very **shallow**, meaning that the well's depth V_0 is extremely small.

(a) WAIT! What kind of sloppy qual problem is this!? It says "V_0 is extremely small" … but *compared to WHAT?!* Well, the problem *is* actually well-posed because there *is only one other energy scale* that is *intrinsic to the problem*. Let's call that intrinsic energy scale β; saying "the well is extremely shallow" means $V_0 << \beta$ i.e. the *dimensionless quantity* V_0 / β can be made *arbitrarily small compared to* 1. So: What is β?
(We aren't given that many quantities, and not that many physical constants are relevant here; it's pretty easy to spot the one combination that has units of energy!)

(b) The state of lowest energy turns out to be a bound state. (You get that information for free.)
Calculate the energy of that bound state, making liberal use of the approximation $V_0 / \beta << 1$.
Show that, astonishingly (!), this bound state survives even in the full limit $V_0 / \beta \to 0$. (Yes, ZERO.)