<div align="center">
You may use anything from the **486 final formula sheets** without derivation
… but do try to see how far you can get on your own on your Desert Island. ☺
You may also use wolframalpha.com or similar tool to evaluate your integrals
after you set the up in a form that can be **directly entered** into such tools.
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Problem 1 : How to (Anti)-Symmetrize   *part(c) adapted from Griffiths 5.7*

Say you have a system of two identical particles, 1 and 2. Let $\hat{P}_{12}$ be the **exchange operator** that swaps particle 1 with particle 2 and vice versa. If we start with a random two-particle wavefunction,

$$\psi(\vec{r}_1, \vec{r}_2),$$

we can always **S**ymmetrize or **A**nti-symmetrize it using these operations:

$$\psi_S(\vec{r}_1, \vec{r}_2) = C_S \left[\psi(\vec{r}_1, \vec{r}_2) + \hat{P}_{12}\, \psi(\vec{r}_1, \vec{r}_2) \right] = C_S \left[\psi(\vec{r}_1, \vec{r}_2) + \psi(\vec{r}_2, \vec{r}_1) \right]$$

$$\psi_A(\vec{r}_1, \vec{r}_2) = C_A \left[\psi(\vec{r}_1, \vec{r}_2) - \hat{P}_{12}\, \psi(\vec{r}_1, \vec{r}_2) \right] = C_A \left[\psi(\vec{r}_1, \vec{r}_2) - \psi(\vec{r}_2, \vec{r}_1) \right]$$

where C_S and C_A are normalization constants to be determined. This simple prescription will obviously work to turn any wavefunction $\psi(\vec{r}_1, \vec{r}_2)$ into an **eigenstate of the exchange operator** $\hat{P}_{12}$. (That's what ψ_S and ψ_A are: eigenstates of the exchange operator with eigenvalues +1 and –1 respectively! ☺)

(a) Easy enough! Now suppose that we have a system of <u>three</u> identical particles instead. Can we construct a similar prescription for (anti)-symmetrizing a random wavefunction $\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$? Sure we can!

Now that we have three particles, we should use more compact notation:

$$\psi_S(\vec{r}_1, \vec{r}_2, \vec{r}_3) = C_S \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boxed{?_S}\, \psi(\vec{r}_i, \vec{r}_j, \vec{r}_k) \text{ for a \textbf{3-boson} system, and}$$

$$\psi_A(\vec{r}_1, \vec{r}_2, \vec{r}_3) = C_A \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boxed{?_A}\, \psi(\vec{r}_i, \vec{r}_j, \vec{r}_k) \text{ for a \textbf{3-fermion} system.}$$

Your job is to figure out what expression goes in the "?" boxes. Each box should contain a term that involves only the **Levi-Civita epsilon** ε_{ijk}, numbers, and/or arithmetic operators (sums, absolute values, powers, etc). If you're not sure what ε_{ijk} is, look it up online. The best way to proceed is to first figure out all the terms you need in your two wavefunctions — there are not that many! —, realizing that $\psi_{S,A}$ should be symmetric / anti-symmetric (i.e stay the same / change sign) under the exchange of <u>ANY TWO</u> (!!) of the particles. Once you know the terms you are trying to build, figuring out what goes in the boxes is straightforward.

(b) In what other context might you have seen the Levi-Civita epsilon before? Most likely it was the following: Let $\vec{a} = (a_x, a_y, a_z)$, $\vec{b} = (b_x, b_y, b_z)$, and $\vec{c} = (c_x, c_y, c_z)$ be three vectors expressed in a Cartesian coordinate system. What operation does the following expression correspond to? i.e., what more familiar notation can you use to do the same thing as this:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\, a_i\, b_j\, c_k$$

There are not that many non-zero terms in this triple-sum … write them out and you will see what it is!

(c) *Griffiths Problem 5.7 goes here*

Problem 2 : Hydrogen Recap & Space × Spin   *adapted from Griffiths 4.55*

We are embarking on a survey of atomic physics. Hydrogen — the simplest of all atoms! — offers us a familiar opportunity to remind ourselves of some important facts: recall that a electron bound in a hydrogen atom has

- a wavefunction with *two* parts:
  a **spatial part** ψ_{nlm} expressed as a function of the coordinates (r,θ,ϕ) and
  a **spin part** $\vec{\chi}$ expressed in the form of a spinor = column-vector with

  basis states $\chi_+ \equiv \chi_{(m_s = +1/2)} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- \equiv \chi_{(m_s = -1/2)} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

- numerous **quantum numbers** to describe all that: $\left| s = \tfrac{1}{2}; n\,l\,m_l\,m_s \right\rangle$, and

- **constraints** on those quantum numbers : $l < n$, $|m_l| \le l$, $|m_s| \le s = \tfrac{1}{2}$.

Second, we will introduce our new tool of <u>angular momentum addition</u> into the system by considering the

- **total angular momentum** $\vec{J} = \vec{L} + \vec{S}$ of the electron.

For the questions below, <u>you can use a table of Clebsch-Gordan coefficients</u> without proof (e.g. from the top-level directory of the website). **IMPORTANT:** C-G tables can be read <u>vertically OR horizontally</u>, i.e.
  you can build one $\left| J, M \right\rangle$ state in terms of several $\left| m_1, m_2 \right\rangle$ states by reading vertically, OR
  you can build one $\left| m_1, m_2 \right\rangle$ state in terms of several $\left| J, M \right\rangle$ states by reading horizontally.

You can prove this easily using the fact that both $\left\{ \left| m_1, m_2 \right\rangle \right\}$ and $\left\{ \left| J, M \right\rangle \right\}$ are complete, <u>orthonormal</u> sets of states that span the space of angular momenta $\vec{j}_1 + \vec{j}_2$ for two given quantum numbers j_1 and j_2.

On to our problem! The electron in a hydrogen atom occupies the combined spin and position state

$$R_{21}\left(\sqrt{1/3}\ Y_1^0\ \chi_+ + \sqrt{2/3}\ Y_1^1\ \chi_- \right)$$

(a) If you measured L^2, what values might you get, and what is the probability of each?

(b) same question for L_z (c) same question for S^2 (d) same question for S_z

(e) same question for J^2 (f) same question for J_z

(g) If you measured the position of the electron, what is the probability density for finding it at (r, θ, ϕ) ?

(h) If you measured both the z component of the spin *and* the distance from the origin (note that these are compatible observables), what is the probability density for finding the electron with spin up at radius r ?

Problem 3 : Two Particles in an ∞-Well Sandbox   *adapted from Griffiths 5.5*

In Discussion 4 Problem 3, you placed two non-interacting particles in a 1D ∞ potential well. If you didn't get to that problem, please work through it first, or at least step carefully through the posted solution. <u>You can use any results from that problem without re-proving them</u>. Note that the discussion problem makes no mention of the particles' spin at all, the problem dealt only with the spatial part of the wavefunction, $\psi(\vec{r}_1, \vec{r}_2)$.
In this problem, we will continue to IGNORE SPIN ENTIRELY until part (c).

(a) Write down the Hamiltonian for two noninteracting identical particles in the infinite square well from Discussion 4 Problem 3. Show explicitly that the <u>fermion ground state</u> $\psi_A(\vec{r}_1, \vec{r}_2)$ you obtained *is*, in fact,

an eigenfunction of the Hamiltonian with the appropriate eigenvalue (i.e. the one you obtained before).

(b) Find the spatial wavefunction(s), energy, and degeneracy of the <u>second excited state</u> for each of these cases:

       (i) two **D**istinguishable particles
       (ii) two identical bosons, i.e. ψ **S**ymmetric under particle exchange
       (iii) two identical fermions, i.e. ψ **A**nti-symmetric under particle exchange

Problem 4 : Reduced Mass in QM   *Griffiths 5.1*

Griffiths Problem 5.1 goes here