

You may use anything from the **486/7 formula sheets** without derivation ... but do try to see how far you can get on your own on your Desert Island. ☺
 You may also use wolframalpha.com or similar tool to evaluate your integrals **after you set the up** in a form that can be **directly entered** into such tools.

Problem 1 : Start in an eigenstate, stay in an eigenstate

Let Q be a Hermitian operator that is associated with an observable and has no explicit dependence on time. We have said that **if Q commutes with the Hamiltonian H** of a system, then the observable associated with Q **is conserved** in the following sense:

If $[Q, H] = 0$ and we place the system in an eigenstate $|q\rangle$ of Q , it will remain in that eigenstate.

If $[Q, H] = 0$, the quantum number associated with the eigenvalues of Q is called a **good quantum number** of the system, as we can meaningfully label the system as “being in the state q ”. If the boxed statement were *not* true, we could only say “the system is in state q at some instant t ”, which would make q a pretty useless label!

So far, we have based the boxed statement on the relation

$$\frac{d\langle \hat{Q} \rangle}{dt} = \frac{1}{i\hbar} \langle [\hat{Q}, \hat{H}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

from 486 ... but $\langle Q \rangle$ is an expectation value – an average – so the stronger boxed statement doesn’t quite follow, does it ... not entirely satisfying! You can prove the boxed statement, go for it! For convenience, if you need to, use the commutator relation $[AB, C] = A[B, C] + [A, C]B$ that you proved last week.

► One possible way to proceed: Place your system in an eigenstate $|q\rangle$ of Q , use our new **time-translation operator** $U(t)$ with **generator** H to write down how it will evolve with time, $|q(t)\rangle$, then see if $|q(t)\rangle$ remains an eigenstate of Q at all times t .

► How NOT to proceed: Do NOT try to show that $|q(t)\rangle = |q\rangle$. Why? Because I can multiply any normalized eigenstate by a phase factor and it will *still* be a normalized eigenstate with the exact same eigenvalue. Example: a free electron starts at $t = 0$ in the momentum eigenstate $|p\rangle = e^{ipx/\hbar}$. Its time evolution is $|p(t)\rangle = e^{ipx/\hbar} e^{-ip^2t/2m\hbar}$. This differs from $|p\rangle$ by the phase factor $e^{-ip^2t/2m\hbar}$ but it clearly still returns the same eigenvalue p when hit with the momentum operator $(-i/\hbar)\partial/\partial x$!

Problem 2 : Relativistic Correction to 1D-SHO™

Qual Classique

At the beginning of §6.3, Griffiths provides a nice little table (Table 6.1) of the approximate size of the primary corrections to the simple central-Coulomb-only-Hamiltonian that we used to get the Bohr wavefunctions and associated energy spectrum of the hydrogen atom. I will leave it to you to read the sections of Chapter 6 that derive these corrections as we have discussed their physical origin already and there are no new tricks to learn. One of them is a relativistic correction. Remarkably, it is of similar magnitude and almost the same form as the spin-orbit correction, so the two are usually bundled together as the **fine structure** correction. It is derived in §6.3.1.

To learn how to do the relativistic correction, let’s do a different one! Calculate the lowest-order (of perturbation theory) relativistic correction to the energy levels E_n of the 1D harmonic oscillator. You may start with Griffiths equation 6.53 as long as you read its derivation (it might appear on an exam).

Commutators in a Nutshell

What are commutators $[A,B]$ good for? We can boil it down to three essentials. Assuming in each case that A and B are Hermitian operators associated with observables A and B (i.e. are operators of physical interest!),

• COMPATIBLE OBSERVABLES

$[A,B] = 0$ means that A and B **share a set of eigenstates** and so are **compatible observables**. Compatibility means that if you start in an eigenstate of A, then measure B, then measure A again, you will discover that you are still in the same exact same eigenstate of A in which you started.

For non-commuting observables, the **Heisenberg uncertainty principle** quantifies just how incompatible they are:

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right|$$

• CONSTANTS OF MOTION

As we just reviewed, observables that have no intrinsic time-dependence and **commute with the Hamiltonian** of a system have constant expectation values and produce **good quantum numbers**.

• LADDER OPERATORS → Quantized Spectra

Problem 3 : Ladder Operators on a Desert Island → A General Treatment

QM textbooks present ladder (a.k.a. raising & lowering) operators on a case-by-case basis. They can instead be presented in a *unified* fashion, which makes it easier to see the “big picture” and to rebuild them from scratch. Let’s use the generic notation Λ for a ladder operator. (The symbol Λ is “L” in greek and it even looks like a ladder!) We don’t know in advance if Λ is raising or lowering so we will leave off the \pm subscript for most of our work. Also, feel free to leave off the hats on the operator symbols as I have done in this paragraph: it is perfectly clear which symbols are operators.

By definition, a **ladder operator Λ is associated with some other operator Q and its eigenstates $|q\rangle$** . Each eigenstate $|q\rangle$ has eigenvalue q and is assumed to be normalized. Now here is the definition of a ladder operator Λ associated with Q :

Words: When Λ operates on a normalized eigenstate $|q\rangle$ of Q , the result is a *different* normalized eigenstate $|q+c\rangle$ of Q .

Math: $\hat{\Lambda}|q\rangle = |q+c\rangle$ where $1 = \langle q|q\rangle = \langle q+c|q+c\rangle$

(a) The definition $\hat{\Lambda}|q\rangle = |q+c\rangle$ of the raising/lowering property is equivalent to the expression

$$\hat{Q}(\Lambda|q\rangle) = \text{something}_1 (\Lambda|\text{something}_2\rangle).$$

What are the two somethings? NOTE: This is the starting definition in most books for “ Λ is a ladder operator”.

(b) Show that $\hat{Q}(\Lambda|q\rangle) = \text{something}_1 (\Lambda|\text{something}_2\rangle)$ is equivalent to this third statement:

$$\boxed{[\hat{Q}, \hat{\Lambda}] = c\hat{\Lambda}} \quad \text{where both sides are assumed to be acting on an eigenstate } |q\rangle \text{ of } \hat{Q} \text{ (see note¹)}$$

¹ This caveat is not necessary if Q is a **Hermitian** operator, e.g. if Q is associated with a physical **observable**.

This is the important one! To prove the equivalence of (a) and (b), I find it easier to work backwards, i.e. start with the commutator relation (b) then derive the “something” relation (a); since each step of your (really short & simple) proof will be an equivalence, you will have trivially proved that the two statements are equivalent.

OKAY! We are now ready to do some desert-island work, i.e. derive things *without* reference materials!
In QM, ladder operators appear mostly in two contexts:

- angular momentum component eigenstates, where $\hat{Q} = \hat{L}_z$ or \hat{S}_z or \hat{J}_z
- energy eigenstates, where $\hat{Q} = \hat{H} =$ the Hamiltonian

(There is at least one other example of a ladder operator but it is rare; if you find others let me know!)

To Be Continued on Homework 9.

Let's start with the first one, $\hat{Q} = \hat{L}_z$, since it is familiar.

(c) With one operation, show that if **Q is a Hermitian operator**, $[\hat{Q}, \hat{\Lambda}] = c \hat{\Lambda}$ implies two things: that c is real and that $[\hat{Q}, \hat{\Lambda}^\dagger] = -c \hat{\Lambda}^\dagger$. This means that Λ and its adjoint Λ^\dagger form a raising & lowering pair!

REMEMBER: We showed that the completely general Inner Product Space notation,

(1) $|\psi\rangle$ = a state in the IPS

(2) \hat{Q} = an operator on a state in the IPS that produces a state in the IPS

(3) $\langle \psi_1 | \psi_2 \rangle$ = the inner product of the , that maps two states onto a scalar

(4) $|\psi\rangle = \sum_{\text{all } n} |e_n\rangle \langle e_n | \psi \rangle$ = the completeness relation showing that all states in the IPS can be written as

a linear combination of orthonormal states $\{|e_n\rangle\}$ in the IPS

Shankar page 203 § 7.4 The Oscillator in the Energy Basis AND 25-26 in §1.6 The Adjoint of an Operator

Case 1 : Angular Momentum $\hat{Q} = \hat{L}_z$

Our task is to *figure out* the ladder operator Λ for $Q = L_z$. QM textbooks start by telling us what Λ_\pm *is*, then go on to demonstrate that it does indeed raise / lower an L_z eigenstate to the one with the next higher / lower eigenvalue $m\hbar$. Let's work backwards instead: Years later, we've completely forgotten the form of Λ_\pm , but we know it exists and we need it! Our starting point will be the **commutator form** of the definition of Λ :

$$[\hat{Q}, \hat{\Lambda}] = c \hat{\Lambda}$$

The reason this form is so useful for finding Λ is that **Λ doesn't commute with Q** . Think through all the commutators you have learned in QM ... MOST of them are zero! There are only *a few* that are non-zero. Once you have a Q , you will pretty quickly figure out what Λ has to be built from!

(Will you be able to derive the boxed relation on a desert island? ABSOLUTELY, and here's why: because you NOW KNOW THAT IT EXISTS, i.e. that a COMMUTATOR FORM of the Ladder-Op Definition exists, so you will know to LOOK FOR IT on your desert island.)

(d) You are trying to find an operator Λ that will satisfy $[L_z, \Lambda] = c \Lambda$ for some scalar c . As noted above, that means Λ doesn't commute with L_z . How many operators can you think of that don't commute with L_z , and what are their commutation relations with L_z ? Write them down. IMPORTANT: Don't go searching very far ... you only need two in this case, and they are the most obvious ones ... there are more operators that don't commute with L_z but there is no need to write them all down if your first two guesses are correct ... ALWAYS TRY THE OBVIOUS / EASY THINGS FIRST.

(e) For the two obvious operators you chose (call them A and B), write down their individual commutation relations with L_z (one line for $[L_z, A]$ and one line for $[L_z, B]$) ... then stare at those two lines and see if you can find some linear combination of A and B from which you can build a Λ where $[L_z, \Lambda] = c \Lambda$. **Serious suggestion:** of course you know the answer, but do try something that *doesn't* work. You learn *way* more from playing with results or strategies that *fail* than ones that succeed! THIS IS WHY PH.D.s TAKE 5-7 YEARS.

(f) You have found a Λ

Problem 4 : Ladder Operators OLD OLD OLD

adapted from Griffiths 4.18

We now continue with the **angular momentum commutator** calculations we were doing in Discussion 13.

We had the fundamental commutators of angular momentum worked out :

$$\boxed{[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0} \quad \text{but} \quad \boxed{[L_x, L_y] = i\hbar L_z, \text{ etc}}$$

We also defined the **ladder operators** L_{\pm} a.k.a. the **raising and lowering operators** :

$$\boxed{L_{\pm} \equiv L_x \pm iL_y} \quad \text{with key commutators} \quad \boxed{[L^2, L_{\pm}] = 0} \quad \text{and} \quad \boxed{[L_z, L_{\pm}] = \pm\hbar L_{\pm}}$$

The main utility of the ladder operators is that they **bump up or down the m quantum number** of an L^2, L_z eigenstate **by 1 unit**. Let's get that in a boxed $L^2 = L_x L_x + L_y L_y + L_z L_z \mp \hbar L_z$ formula: if ψ_l^m is an eigenstate of L^2 and L_z with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$ respectively, then

$$\boxed{L_{\pm} \psi_l^m = \pm\hbar \psi_l^{m\pm 1}} \quad \Leftarrow \quad L_+ \text{ raises the } m \text{ value by 1 and } L_- \text{ lowers it by 1.}$$

That's how far we got in Discussion 13. Onwards!

(a) Clearly, $L^2 = L_x^2 + L_y^2 + L_z^2$. Prove that . (You will need it in the next part).

(b)

last week : 1D SHO setup

(b) Consider the 1D-SHO TM with Hamiltonian

$$H(x) = \frac{\hbar\omega}{2} \left(\xi^2 - \frac{d^2}{d\xi^2} \right) \quad \text{where} \quad \xi \equiv \frac{x}{x_0} \quad \text{and} \quad x_0 \equiv \sqrt{\frac{\hbar}{m\omega}} \quad \text{and } m, \omega \text{ are known constants.}$$

For convenience, assume we are working in units where $x_0 = 1$ (which is always possible!) ☺

last week : conclusion missing

(d) Now ask yourself this immense question:

What are the good quantum numbers for atomic electrons?

Hmm! We've seen tables of elements where **spectroscopic notation** ^{2S+1} \mathcal{L}_J is used to specify the ground state.

To understand why this notation is used, you must calculate 6 commutators:

$$(i) [\vec{L} \cdot \vec{S}, \vec{L}] \quad (ii) [\vec{L} \cdot \vec{S}, \vec{S}] \quad (iii) [\vec{L} \cdot \vec{S}, \vec{J}] \quad (iv) [\vec{L} \cdot \vec{S}, L^2] \quad (v) [\vec{L} \cdot \vec{S}, S^2] \quad (vi) [\vec{L} \cdot \vec{S}, J^2]$$

Please do not repeat algebra unnecessarily! Use symmetries / similarities between these various objects to argue things like “since A,B have the same relations as C,D, I can use my result for A,B directly for C,D too”.