Phys 487 Discussion 1 – Angular Momentum & Commutator Algebra

When we first studied angular momentum, we combined the relation $\vec{L} = \vec{r} \times \vec{p}$ and the QM operators

$$\hat{\vec{r}} = \vec{r} = (x, y, z) \qquad \& \qquad \hat{\vec{p}} = \frac{\hbar}{i}\vec{\nabla} = \frac{\hbar}{i}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

to obtain the operator $\hat{\vec{L}}$ for orbital angular momentum. In particular, we found these commutation relations :

$$\left[\hat{L}^2, \hat{L}_i\right] = 0 \text{ where } i = x, y, \text{ or } z \qquad \text{and} \qquad \left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z \text{ + cyclic permutations thereof.}$$

In words, $L^2 \equiv \vec{L} \cdot \vec{L}$ commutes with each component of $\vec{L}$, but the components don't commute with each other.

That was for **orbital** angular momentum (OAM). We then encountered experimental evidence for another type of angular momentum: **spin** angular momentum. We called this $\vec{S}$. It is an *intrinsic* property of elementary particles, like mass or charge, and behaves a great deal like the *rotational* angular momentum of classical objects about their center-of-mass. To work with spin, we literally *copied* all the relations we derived for OAM, notably those commutators:

$$\left[\hat{S}^2, \hat{S}_i\right] = 0 \text{ where } i = x, y, \text{ or } z \qquad \text{and} \qquad \left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z \text{ + cyclic permutations thereof.}$$

Experiment indicates that spin really is an angular momentum like OAM in that the two *add together*. We introduced the letter $\vec{J}$ to describe two things :

- <u>Total</u> angular momentum, as in the *sum* of OAM and spin : $\vec{J} = \vec{L} + \vec{S}$
- <u>Any</u> angular momentum: *j* or *J* can stand for an OAM, or a spin, or the sum of 2 spins and an OAM, … Since our description of spin is *copied* from our description of OAM, we need some letter that can generically refer to either one!

So finally, the commutators for quantum angular momentum – spin, OAM, or their sum – are :

$$\left[\hat{J}^2, \hat{J}_i\right] = 0 \text{ where } i = x, y, \text{ or } z \qquad \text{and} \qquad \left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z \text{ + cyclic permutations thereof.}$$

Here is a problem from a compendium of Ph.D. qualifying exam problems that shows just how powerful those angular momentum relations are :

Problem 1 : Algebra of Angular Momentum *Qual Problem (Stony Brook)*

Given the commutator algebra

$$\left[J_1, J_2\right] = iJ_3 \qquad \left[J_2, J_3\right] = iJ_1 \qquad \left[J_3, J_1\right] = iJ_2$$

(a) Show that $J^2 \equiv J_1^2 + J_2^2 + J_3^2$ commutes with J_3.

(b) Derive the spectrum of $\left\{J^2, J_3\right\}$ from the commutation relations.

▶ Part (a) is no problem, a little refresher on working with commutators.
▶ Part (b), however, is BIG. Since this is your first time seeing it, some hints are coming up on the next page. The biggest hint you need is LADDER OPERATORS, i.e. these guys: $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. Remember them … ?

Important: *Everything* in this qual problem has to be *explicitly derived* from nothing but the three given commutation relations. That is, of course, what makes this problem *so awesome*. :-D
Also, since this is your first time working this problem, you can consult the 486 formula sheet on our website as a memory-assist … but remember that you will not have much / any of a formula sheet on the qual exam (the UIUC qual sheet is on our website if you're interested). That, of course, makes the qual *awesome*. :-D

Now for some guidance as to how to proceed with part (b). *with some hints in footnote*[1]

[STEP 1] **Introduce labels for the eigenvalues** that you are trying to find.

What are you trying to find again? → the spectrum of the operators J^2 and J_3. The spectrum of an operator means all the allowed = measurable values of the associated observable, which in QM means all the eigenvalues of the operator. We can't proceed before we have labels for all of our knowns and unknowns, so we need a symbol for the eigenvalues of J^2 and another one for the eigenvalues of J_3. We could use something generic like a and b :

$$J^2|a,b\rangle = a\,|a,b\rangle \quad \text{and} \quad J_3|a,b\rangle = b\,|a,b\rangle\,.$$

This will work … but we will make our life *much* easier if we can guess what the answer is going to be.
We derived the spectra for the *orbital* angular operators L^2 and L_z … do you remember the form of their eigenvalues … ? Take your best shot at ideal labels for the eigenvalues of J^2 and J_3, then check the footnote.

[STEP 2] **Introduce and derive the ladder operators for** $\hat{J}_3$

Here, you really need to remember the form of the ladder operators $\hat{J}_\pm$ that step-up and step-down the eigenvalues of $\hat{J}_z$. Their form is $\hat{J}_\pm = \hat{J}_x \pm i\,\hat{J}_y$. Now it is up to you to prove that they *are* the ladder operators of $\hat{J}_z$ (and translate *xyz* into 123). What formula must you prove to show this? (footnote has sequence of hints)

[STEP 3] **Normalize the ladder operators for** $\hat{J}_3$

You just showed that $\hat{J}_\pm|j,m\rangle \sim |j,m\pm 1\rangle$, which means $\hat{J}_\pm|j,m\rangle = c_{j,m,\pm}|j,m\pm 1\rangle$ for some normalization constant $c_{j,m,\pm}$. You derived this constant in Homework 12 last year; here's the result so you can keep going

$$\hat{J}_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\;|j,m\pm 1\rangle$$

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[1] **STEP 1** : introduce j and m as labels for the eigenstates of J^2 and J_3, i.e. let the eigenstates be $|j,m\rangle$. For the eigenvalues, use our prior knowledge to guess these forms: $J^2|j,m\rangle = j(j+1)|j,m\rangle$ and $J_3|j,m\rangle = m|j,m\rangle$. Why does this help? It might *not* help, but for *OAM* at least, we know that the eigenvalues of L^2 and L_z are $l(l+1)\hbar^2$ and $m\hbar$ with m and l = integers. We will see if this problem also ends up with eigenvalues of the form $j(j+1)$ and m for integer-only values of j and m …

STEP 2 : You must show that $\hat{J}_\pm|j,m\rangle \sim |j, m\pm\Delta m\rangle$ for some Δm. In words, when you hit an eigenstate of J_3 with $J_\pm$, you get another eigenstate of J_3, just with a larger or smaller eigenvalue m. And to show that, you must show ….

STEP 2′ : $\hat{J}_3\left(\hat{J}_\pm|j,m\rangle\right) = (m\pm\Delta m)\left(\hat{J}_\pm|j,m\rangle\right)$ for some Δm … now we notice the product $\hat{J}_3\hat{J}_\pm$ in there; what would be useful?

STEP 2″ : A product of two operators is involved? Hoho! I bet the *commutator* $\left[\hat{J}_3, \hat{J}_\pm\right]$ of those operators would help!
→ now we are good to go for STEP 2 : calculate the commutator $\left[\hat{J}_3, \hat{J}_\pm\right]$, then prove STEP 2′, which proves STEP 2.

STEP 2‴ : In principle, you must also show that $\hat{J}^2\left(\hat{J}_\pm|j,m\rangle\right) = j(j+1)\left(\hat{J}_\pm|j,m\rangle\right)$, i.e. that your proposed ladder operators do *not* change the eigenvalue $j(j+1)$ of J^2, they only change the eigenvalue m of J_3. The method is the same.

[STEP 4] **Restrict the allowed values of *m*** *further hints in footnote*[2]

Time to start constraining the eigenvalues m and $j(j+1)$ to find the allowed spectrum. First, we can easily constrain m relative to j because we know the physics behind the operators:

- J_3 is one component of $\vec{J}$, while
- J^2 is the magnitude² of $\vec{J}$, with $J^2 \equiv J_1^2 + J_2^2 + J_3^2$ from part (a).

Clearly the (eigenvalue)² of J_3 can never be bigger that the eigenvalue of J^2! For a given value of j, what values of m are allowed?

[STEP 5] **Restrict the allowed values of *j***

This is the last part! Do you recall how you used the ladder operators $\hat{a}_\pm$ for the harmonic oscillator Hamiltonian to produce its energy spectrum? The basic idea is "if you know one eigen-energy, apply the step-up operator and you'll get another one that's higher! … or apply the step-down operator and you'll get another energy that's lower! …" all of which produces a lovely ladder of energy eigenvalues. But can we continue forever up the ladder, or forever down the ladder, with repeated applications of $\hat{a}_+$ or $\hat{a}_-$ … ?

That may be enough of a hint for you to finish the problem and find all the allowed values of j. Otherwise, a sequence of hints is in the footnote.

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[2] **STEP 4** : $m^2 \le j(j+1)$, directly from $J^2 \equiv J_1^2 + J_2^2 + J_3^2$ and the positivity of $J_1^2 + J_2^2$.

STEP 5 : The ladder operators in this problem are $J_\pm$ … what eigenvalue do they increase / decrease?

STEP 5′ : They affect m, increasing / decreasing it by 1 … now: can you keep increasing and/or decreasing m forever?

STEP 5″ : No: the ladder operators do *not* affect j, so we cannot keep increasing m forever with repeated applications of J_+ or we will violate the limit $m^2 \le j(j+1)$ from STEP 4. Same issue with repeated applications of J_-. Now, can you write down a mathematical expression that says "too many applications of J_+ or J_- to a given state $|j,m\rangle$ are not allowed" ?

STEP 5‴ : $\hat{J}_+|j, m_{\rm max}\rangle = 0$ and $\hat{J}_-|j, m_{\rm min}\rangle = 0$ … so what *are* the min and max values of m for a given j? STEP 4 provides an upper limit, but not a value … instead you must use an earlier formula …

STEP 5⁗ : Evaluate $\hat{J}_+|j, m_{\rm max}\rangle = 0$ and $\hat{J}_-|j, m_{\rm min}\rangle = 0$ using the normalized formula from STEP 3 → the result is $m_{\rm max} = j$, $m_{\rm min} = -j$. Great! The next step is very cool: we must remember the *ladder* and realize that we can get from state $|j,m_{\rm min}\rangle$ to state $|j,m_{\rm max}\rangle$ by repeated application of $\hat{J}_+$. That tells us something about the quantity $(m_{\rm max} - m_{\rm min})$ ...

STEP 5′′′′′ : The ladder takes us from $m_{\rm min}$ to $m_{\rm max}$ via this operation : $(\hat{J}_+)^p|j, m_{\rm min}\rangle \sim |j, m_{\rm max}\rangle$ where p is an integer : it is a *count* of the number of step-up operations needed. Since each application of J_+ changes m by +1, $m_{\rm max} - m_{\rm min} = p$ = an integer. Now you have everything you need to determine the spectrum of possible values of j …

FINAL RESULT : $j = p/2$ and is therefore restricted to integer or half-integer values (0, ½, 1, 3/2, 2, 5/2, …)