Phys 487 Discussion 3 – Hamiltonians as Matrices and Good Quantum Numbers

Problem 1: Symmetries and Good Quantum Numbers.

The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

where ω is a positive real number. Two other observables, A and B, in the same basis are represented as the matrices

$$A = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where λ and μ are positive real numbers.

(a) Compute the commutators $[H, A]$ and $[H, B]$. If the commutator with the Hamiltonian is zero, then the observable is called a symmetry of the system (or an integral of motion). Are either A or B symmetries?

(b) Suppose the system starts out at time $t = 0$ in a generic normalized state $|\psi(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Compute the expectation values of A and B at $t = 0$.

(c) Find the time-evolved state $|\psi(t)\rangle$. Compute the expectation values of A and B at time $t > 0$. A key property of symmetries is that their expectation values stay constant over time. You should see that this is true for one of the operators and not the other.

(d) The observable B has an eigenvector $|\mu\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ with eigenvalue μ. Show that if $|\psi(0)\rangle = |\mu\rangle$ at time $t = 0$, then the time-evolved state $|\psi(t)\rangle \propto |\mu\rangle$ for all times $t > 0$. Since B is a symmetry, its eigenvalues are *good quantum numbers*. If a state is an eigenstate of a symmetry, it will stay an eigenstate of that symmetry over time and will always have the same quantum number.

Problem 2: Electron in an External Field (Griffiths 4.33).

An electron is at rest in an oscillating external magnetic field that points in the z-direction

$$\boldsymbol{B} = B_0 \cos(\omega t)\,\hat{z}$$

where B_0 and ω are constants. The magnetic field couples to the spin of the electron so that the Hamiltonian for this system is

$$H = -\gamma \boldsymbol{B} \cdot \boldsymbol{S}$$

where γ is the gyromagnetic ratio and $\boldsymbol{S}$ is a vector of spin angular momentum operators.

(a) Write down the Hamiltonian in terms of the S_z operator.

(b) Remember that the electron is a spin-1/2 particle. The electron's spin can be in the state $|s, m_s\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle = |\uparrow\rangle$ or $|s, m_s\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = |\downarrow\rangle$ or any linear combination of these two states. Write down the Hamiltonian as a 2×2 matrix in the basis of states $\{|\uparrow\rangle, |\downarrow\rangle\}$.

(c) Suppose that the electron is initially pointing in the $+x$-direction. That means that it starts in the $+\hbar/2$ eigenstate of S_x, which is $|\psi(0)\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. Determine the time-evolved state $|\psi(t)\rangle$ under the given Hamiltonian. ***Hint:*** *Since the Hamiltonian is* ***time-dependent****, you need to directly solve the time-dependent Schrodinger equation. In this case, it ends up being simple to do so.*

Problem 3: Spin-Orbit Coupling and Good Quantum Numbers.

Taking into account the effect of spin-orbit coupling, the Hamiltonian of an electron in the hydrogen atom is

$$H = H_0 + \beta \boldsymbol{L} \cdot \boldsymbol{S}$$

where β is a positive real number, H_0 is the unperturbed Hamiltonian that involves kinetic and potential energy, and the second term is the perturbation due to the spin-orbit interaction.

Notice that there are two sets of angular momentum operators, $\{L_1, L_2, L_3\}$ and $\{S_1, S_2, S_3\}$, which both obey the canonical angular momentum commutation relations

$$[L_1, L_2] = iL_3 \quad \text{(and cyclic permutations)}$$

$$[S_1, S_2] = iS_3 \quad \text{(and cyclic permutations)}$$

and also commute with one another so that $[L_i, S_j] = 0$ for all $i, j \in \{1,2,3\}$.

The observables L^2, S^2, L_3, and S_3 all commute with H_0, which means that their eigenvalues, $l(l+1)$, $s(s+1)$, m_l, and m_s, are all *good quantum numbers* in the unperturbed system. However, this is not true after the perturbation is added.

(a) Show that $[H, L_3]$ and $[H, S_3]$ are not zero. The eigenvalues of L_3 and S_3, m_l and m_s, are no longer good quantum numbers.

(b) Show that $[H, L^2] = [H, S^2] = 0$. The eigenvalues of L^2 and S^2, $l(l+1)$ and $s(s+1)$, remain as good quantum numbers.

Not part of the problem, but important info: With a little more work, you can also show that the observables $J^2 = (\boldsymbol{L} + \boldsymbol{S})^2$ and $J_3 = L_3 + S_3$, which correspond to the total orbital plus spin angular momentum, also commute with H. Their eigenvalues, $j(j+1)$ and m_j, are good quantum numbers in the presence of the spin-orbit interaction.