Phys 487 Discussion 8 – The WKB Approximation

ψ_A = wavefunction for case $E > V$ = classically-**Allowed** region where E is **Above** the potential

ψ_B = wavefunction for case $E < V$ = classically-**Bonkers** regions where E is **Below** the potential

WKB solution forms in regions where they're valid (extensively discussed) : with $p(x) \equiv \sqrt{2m(E-V(x))}$,

$$\psi_{\mathbf{A}}(x) = \frac{A}{\sqrt{p(x)}} \exp\left[\pm i \int^x \frac{p(x')}{\hbar} dx'\right] \qquad \text{and} \qquad \psi_{\mathbf{B}}(x) = \frac{B}{\sqrt{|p(x)|}} \exp\left[\pm \int^x \frac{|p(x')|}{\hbar} dx'\right]$$

Connection formulae to match WKB solutions at **turning points** $x = a$ (i.e. points where $E = V(a)$) :
using the shorthand $\int k \equiv \int p(x')/\hbar \; dx'$,

barrier on the LEFT :

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{C}{\sqrt{|p|}} \exp\left[-\int_x^a |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_a^x k - \frac{\pi}{4}\right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{D}{\sqrt{|p|}} \exp\left[+\int_x^a |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_a^x k + \frac{\pi}{4}\right]$$

barrier on the RIGHT :

$$\psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_x^a k - \frac{\pi}{4}\right] \quad \text{matches to} \quad \psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{C}{\sqrt{|p|}} \exp\left[-\int_a^x |k|\right]$$

$$\psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_x^a k + \frac{\pi}{4}\right] \quad \text{matches to} \quad \psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{D}{\sqrt{|p|}} \exp\left[+\int_a^x |k|\right]$$

NOTE 1 : The "barrier on the **right**" formulae are **IDENTICAL** to the "barrier on the **left**" ones except for the **reversed order of the integral bounds**.

NOTE 2 : In ALL cases, the lower bound is smaller than the upper bound.

NOTE 3 : HOW TO USE THE CONNECTION FORMULAE

- The connection formulae *replace* the standard boundary conditions "ψ continuous & ψ' continuous" at turning points $x=a$ where <u>$V(x)$ has a finite slope</u>.
- If, instead, <u>$V(x)$ is discontinuous</u> at a turning point (e.g. a step) then you just use the standard boundary conditions "ψ continuous & ψ' continuous".

Why? If a turning point occurs at a place where $V(x)$ has a discontinuous step, then there is no finite region around the turning point where $E \approx V(x)$, and so *there is no transition region where the WKB approximation fails*. The WKB solution forms work right up to the discontinuity on both sides, so there is no need for Airy functions and the resulting connection formulae. You have a WKB-approximated waveform to the left of the turning point, and another WKB waveform to the right, and you patch them together as you have always done.

Problem 1 : 1D SHO ™ *Griffiths 8.7*[1]

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[1] Q1(b) **step 3**: The outer regions B and C go off to x=–∞ and x=+∞ respectively. The wavefunctions in these regions can be restricted to only the exp[+∫…] for the negative-x region and only exp[–∫…] for the positive-x region.
step 5: you must make those two forms for ψ_A from step 4 *consistent with each other,* i.e. *equal* to each other. This consistency requirement will provide the quantization condition on the energy and give you the result you know and love for the 1D SHO!

(a) Before you do anything else, make a small sketch next to the "barrier on the LEFT" formulae above and another one next to the "barrier on the RIGHT" formulae that indicate what these situations refer to. On each sketch $V(x)$ with a solid line, E with a horizontal dashed line, and mark the turning point $x=a$ at the spot where they cross ($E = V(a)$ defines a turning point a).

(b) Use the WKB approximation to find the allowed energies of the harmonic oscillator.

PROCEDURE:

1. Plot the potential $V(x)$, draw the energy E as a dashed line, and mark the turning point(s) with dots.
2. Label the regions that are separated by the turning points. Here, you have three regions. One is classically **A**llowed, label that region $\psi_\mathbf{A}$. The other two regions are classically **B**anned, label those ψ_B and ψ_C.
3. In general, the wavefunction in each region will be a superposition of two forms, such as
$$\psi_\mathbf{A} = \frac{a_1}{\sqrt{p(x)}}\exp\left[+i\int^x k\right] + \frac{a_2}{\sqrt{p(x)}}\exp\left[-i\int^x k\right] \quad \text{or} \quad \psi_\mathbf{A} = \frac{a_1}{\sqrt{p(x)}}\cos\left[\int^x k - \frac{\pi}{4}\right] + \frac{a_2}{\sqrt{p(x)}}\cos\left[\int^x k + \frac{\pi}{4}\right]$$
in a classically allowed region, or
$$\psi_\mathbf{B} = \frac{b_1}{\sqrt{|p(x)|}}\exp\left[+\int^x |k|\right] + \frac{b_2}{\sqrt{|p(x)|}}\exp\left[-\int^x |k|\right]$$
in a classically banned region. But before writing down all these forms with all their free parameters, ask yourself: are there any regions where I can further restrict the wavefunction? In this example, there are two such regions. If you are not sure, have a look at the footnote! Write down the wavefunction forms in those two regions, and leave ψ_A alone for the moment.
4. Now apply boundary conditions to connect the solution forms from the different regions to each other. Read NOTE 3 on the first page to figure out what boundary conditions must be used: the usual ones (ψ and ψ' continuous, or $\psi=0$ at $V=\infty$ walls) or the new WKB connection formulae. In this problem, you will see that you will get *two different forms* for ψ_A, resulting directly from the connection formulae.
5. Finally solve for any free parameters that you need to solve for. In this problem, all you want is the spectrum E_n of allowed energies … what do you have left to do? See the footnote for a hint.

Problem 2 : Bouncing Ball *Qual Problem / hacked bits of Griffiths 8.5 & 8.6*[2]

Consider the QM analog of the classical problem of a ball of mass m bouncing elastically on the floor.

(a) What is the potential energy, as a function of height x above the floor? (For negative x, the potential is *infinite* — the ball can't get there at all.) Sketch this $V(x)$ so you know what to do next!

(b) Using the WKB approximation, find the allowed energies, E_n, of the bouncing ball in terms of m, g, and $\hbar$.

(c) Give the ball a mass of $m = 0.1$ kg = 100 g. (A regulation tennis ball has a mass around 60 g, so let's call it a tennis ball.) What are the energies of the ground state and first-excited state of the tennis ball, and what maximum height does the tennis ball reach in each of these states?

(d) What values do you get for question (c) if you replace the tennis ball with a neutron?

FYI: If we replaced it with an electron – a charged particle of extremely tiny mass – we would have to design *insanely perfect shielding* to shield the electron from *all* stray electromagnetic fields, or we would never see the tiny effect of the gravitational field. So … on page 333, Griffiths has a footnote with a reference to an actual experiment that actually measured the quantum effects of a neutron "bouncing" in gravity. Oy!

(e) What value of the quantum number n is required for the 100 g tennis ball to reach a max height of 1m?

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[2] Q2: (a) $V(x)=mgx$ for $x>0$; $V(x)=\infty$ for $x\leq 0$ (b) $E_n = [\,3/2\,(n-1/4)\,\pi g\hbar\,]^{2/3}\,(\,m/2\,)^{1/3}$ (c) 9×10^{-23} m, 1.6×10^{-22} m
(d) 14 μm, 24 μm (e) $n \approx 3 \times 10^{31}$ heee