

You may use anything from the **486/7 formula sheets** without derivation ... but do try to see how far you can get on your own on your Desert Island. ☺  
 You may also use **wolframalpha.com** or similar tool to evaluate your integrals **after you set the up** in a form that can be **directly entered** into such tools.

**Problem 1 : Variational Principle : 1D SHO**

*adapted from Griffiths 7.12*

(a) Find the best bound on the ground state energy,  $E_{gs}$ , of the 1D SHO <sup>TM</sup> using a trial wavefunction of the form

$$\psi(x) = \frac{A}{(x^2 + b^2)^n}.$$

for arbitrary  $n$  (i.e.  $n$  is supposed to be a given value, not a free parameter, but it is not restricted to integers). Many hints now follow to help you manage this calculation, please read them first:

► Find the optimal value of the variational parameter  $b$  but **DO NOT** find the optimal value of the other variational parameter  $n$ . Leave  $n$  in your answers, as part (c) addresses the variation of  $n$ .

► Lots of integration to do, please DO use **wolframalpha**. You will need the following integral repeatedly, and wolframalpha annoyingly won't do it, so please use this formula freely :

$$\int_0^\infty \frac{x^k}{(x^2 + b^2)^a} dx = \frac{1}{2b^{2a-k-1}} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{2a-k-1}{2})}{\Gamma(a)}$$

Also check out **integral-calculator.com**, which *can* do the previous integral. It comes out in terms of beta functions  $B(x,y)$  which are “Euler beta functions” if you want to pick them out from the many other “beta functions” in math and physics.

► Lots of **Γ functions** about. For this question all you need to evaluate are ratios of Γ functions, and in that context this well-known relation is all you need:

$$\Gamma(n) = (n-1)! \quad \text{or the recursive version, } \Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \dots$$

Don't worry about the values of  $\Gamma(1/2)$  and such, you can look all of that up if you want. Just keep the  $\Gamma$  functions as is until *all* of your integrals are done, then you will see that many of them cancel, and all you need are *ratios* of  $\Gamma$  functions *whose arguments are separated by integers*. Those are all doable with  $\Gamma(n)=(n-1)!$  For example,  $\Gamma(9/2) / \Gamma(5/2) = (7/2)! / (3/2)! = 7/2 \cdot 5/2 = 35/4$ . If you look in the checkpoints file, you will find an intermediate checkpoint that shows the expression you should get after all the gamma functions are gone.

(b) Find the lowest upper bound on the first excited state of the harmonic oscillator using a trial wavefunction of the form

$$\psi(x) = \frac{Bx}{(x^2 + b^2)^n}.$$

To save you from having to do the same integration as you did in part (a), **start from here** :

$$\langle T \rangle = \frac{3\hbar^2}{4mb^2} \frac{n(4n-3)}{(2n+1)} \quad \text{and} \quad \langle V \rangle = \frac{3}{2} \frac{m\omega^2 b^2}{(4n-5)}$$

when the new trial wavefunction is used.

(b') Why does the trial wavefunction you used in part (b) give you an upper limit on the first-excited state whereas the trial function from part (a) gave you an upper limit on the ground state? A brief but clear qualitative answer is what we're looking for, no calculations necessary.

(c) It is clear by inspection of your part (a) and (b) results that the variational-method energies of the ground state and first excited state reach the exact SHO values in the limit  $n \rightarrow \infty$ . (If it is not obvious by inspection, please ask!) Why do we get such perfect results? To find out, show that the trial wavefunctions from parts (a) and (b) approach the exact SHO wavefunctions in the limit  $n \rightarrow \infty$ .

► You will need **Stirling's approximation**:  $\ln(z!) \approx z \ln(z) - z$  for  $z \gg 1$ . The exponential of that relation will allow you to work with the factorial  $z!$  in the limit  $z \rightarrow \infty$ .

## Problem 2 : On the nature of Helium, Screening, and Effective Charge

Griffiths' section §7.2 goes through the most CLASSIC application of the variational principle: finding the ground state energy of the helium atom including the mutual Coulomb repulsion between the two electrons. When we discussed **screening** in our atomic structure section, we advertised that we would come back to the concept of an effective nuclear charge  $< Z$  and calculate one when we had more tools. Well here we are!

First, here is a summary of Griffiths §7.2. The Hamiltonian used is :

$$H(\vec{r}_1, \vec{r}_2) = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{e^2}{4\pi\epsilon_0} \left( \frac{2}{r_1} + \frac{2}{r_2} - \frac{1}{|\vec{r}_1 - \vec{r}_2|} \right)$$

No spin-dependent forces here, just the Coulomb attraction of each electron to the  $Z_{\text{He}}=2$  nucleus and the electron-electron repulsion term (which we call  $V_{ee}$  for short). The trial wavefunction used for this classic problem is essentially the product of two hydrogen wavefunctions, one for each of helium's electrons:

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{Z^3}{\pi a_0^3} e^{-Z(r_1+r_2)/a_0} \quad \text{where } a_0 \text{ is the Bohr radius (see 486 formula sheet).}$$

“Wait”, you remark, “if  $a_0$  is the Bohr radius, that trial wavefunction has no variable parameter, I thought we were always supposed to include at least one in order to minimize our ground-state energy as much as possible?” You are correct ... and there *is* a variable parameter: it is  $Z$ . “But  $Z = 2$  for Helium!” you exclaim, “I can't play around with the truth!” True ... but the idea behind this trial wavefunction is to introduce an **effective  $Z$**  that will come out *less* than  $Z_{\text{He}} = 2$  because the effect of the  $V_{ee}$  term is to partially **screen the nuclear charge**. To understand, imagine you are one of the two electrons and that your name is Ivan. You are attracted to the positively-charged  $Z_{\text{He}} = 2$  nucleus, which sits at the center of your probabilistic cloud of spherically-symmetric  $s$ -shell existence ... but you are repelled by the other electron, named Juan, which is probabilistically splattered through space the same way you are. If you think about Juan using your mastery of Gauss' Law from PHYS 212 (footnote available<sup>1</sup>), you will realize the following: if you, Ivan, are momentarily located a distance  $r_{\text{ivan}}$  away from the nucleus, the *only* portion of Juan's probability cloud that affects you is the part that is closer to the nucleus than you are, i.e. the part with  $r_{\text{juan}} < r_{\text{ivan}}$ . Juan is negatively charged, as are you, so the electric field you see at your location is due to attraction from the charge  $+Z_{\text{He}}$  nucleus & repulsion from a portion of Juan's charge  $-1$  cloud. Thus, Juan **screens** part of the  $+Z_{\text{He}}$  nuclear charge from you. Our

<sup>1</sup> From Gauss' Law: the electric field  $E(r)$  at a distance  $r$  from the center of a spherically symmetric charge distribution is  $Q_{\text{enc}}(r)/4\pi\epsilon_0 r^2$  where  $Q_{\text{enc}}(r)$  stands for the total charge **enclosed within the distance  $r$** . None of the charge outside the radius  $r$  affects the electric field  $E(r)$  at all. Another way to put this is that a spherically symmetric **shell** of charge produces **zero E-field** everywhere inside the empty spherical hole in the middle. That is a fairly AMAZING result from Gauss' Law, actually ... think of all the cancellations of all the little field vectors at every point in the empty core that has to occur ... symmetry, dude, wow.

hope is that introducing an effective = screened charge  $Z$  as a variational parameter will provide a good approximation to the full effect of the  $V_{ee}$  term.

As you see in Griffiths, the best values obtained for the He ground state using the “(hydrogen)<sup>2</sup>-with-screening” trial function are  $Z = 1.69$  (which is less than 2, so yes, the helium nucleus is screened in this model) and  $E_{gs} = -77.5$  eV. This is absurdly close to the experimental value of **-79 eV**. You may recall that in a previous homework, you used 1<sup>st</sup> order perturbation theory to calculate the very same thing – the effect of  $V_{ee}$  on the helium ground state energy – and you obtained  $E_{gs} = -75$  eV. That’s not quite as good, but of course with perturbation theory, you can keep going, to 2<sup>nd</sup> order, 3<sup>rd</sup> order, etc with enough time/energy/processors.

(a) The “(hydrogen)<sup>2</sup>-with-screening” trial function does NOT do such a good job in obtaining the ground state of the **H<sup>-</sup> ion**, i.e. a hydrogen nucleus surrounded by two electrons in a closed  $1s^2$  shell. Using the trial wavefunction described above, calculate the approximate ground state energy of the H<sup>-</sup> ion with  $V_{ee}$  repulsion included.

► To **SHORTEN** this problem, you may use any result from Griffiths §7.2, just give the equation number.

► FYI: It is very common in atomic calculations, such as this one, to need  $\langle 1/r \rangle$  and/or  $\langle 1/r^2 \rangle$  for hydrogenic wavefunctions  $\psi_{nlm}$ . The answers are equations 6.55 and 6.56 in Griffiths, and you are free to use them, but I do want you to know that there is a very fast way to get the first one. You use the super-useful Virial Theorem, which is true in both QM and CM:  $\langle V \rangle = -2\langle T \rangle$  for a particle bound in a  $1/r$  potential. Since you know  $\langle E \rangle = \langle T + V \rangle$  for the hydrogenic wavefunctions (well, it’s on your formula sheet) and  $V \sim 1/r$ ,  $\langle 1/r \rangle$  is easily obtained! This is problem 6.12 in Griffiths; the calculation of  $\langle 1/r^2 \rangle$  is addressed in problem 6.33 & 6.32, which is a bit of a project.

(b) Let’s leave the H<sup>-</sup> ion alone for a moment and jargon-bust the word **ionization**. It means “the freeing of an electron from an atomic bound state”. Remember that to **free** a bound particle means to elevate its total energy from a negative value (its bound-state energy) to ZERO, at which point it is able to reach  $\infty$  = a place free of influence from any other forces ( $V=0$ ) with just enough kinetic energy to be physical ( $T=0$ ).

Consider the **He<sup>+</sup> ion** = a helium atom with one electron removed. As noted above, a He atom has ground state energy -79 eV; calculate the **ionization energy** required to free one electron from a ground-state He atom and turn it into a ground-state He<sup>+</sup> ion.

► Massive hints: (i) consult the 486 formula sheet, and (ii) this is a very short problem.

(c) Back to the H<sup>-</sup> ion. Compare the ground state energy of the H<sup>-</sup> ion that you got in part (a) to the ground state energy of the H atom that you got in 486. If you do not *instantly* know the ground-state energy of hydrogen, you *must* memorize it. It is -13.6 eV. It is one of the handful of numbers that all physicists have committed to memory. It even has a name, the **Rydberg energy**. Griffiths calls it  $E_1$  in this section. Suppose an astronomer tells you that the H<sup>-</sup> ion is totally stable, and is called “hydride”, and is commonly found in the atmosphere of stars. Does your calculation from (a) support this information? Why or why not?

**FYI:** The solution to the wonky value of the H<sup>-</sup> energy is in problem 7.18. Do read the problem, the concept is very nice! But the solution requires many integrals, and you did so many in problem 1 ... we’ll skip it.

### **Problem 3 : WKB Bouncing Ball = Discussion 8 Question 2**      *Qual Problem / bits of Griffiths 8.5 & 8.6*

Consider the QM analog of the classical problem of a ball of mass  $m$  bouncing elastically on the floor.

(a) What is the potential energy, as a function of height  $x$  above the floor? (For negative  $x$ , the potential is *infinite* — the ball can’t get there at all.) Sketch this  $V(x)$  so you know what to do next!

(b) Using the WKB approximation, find the allowed energies,  $E_n$ , of the bouncing ball in terms of  $m$ ,  $g$ , and  $\hbar$ .

(c) Give the ball a mass of  $m = 0.1$  kg = 100 g. (A regulation tennis ball has a mass around 60 g, so let’s call it a tennis ball.) What are the energies of the ground state and first-excited state of the tennis ball, and what maximum height does the tennis ball reach in each of these states?

(d) What values do you get for question (c) if you replace the tennis ball with a neutron?

**FYI:** If we replaced it with an electron – a charged particle of extremely tiny mass – we would have to design *insanely perfect shielding* to shield the electron from *all* stray electromagnetic fields, or we would never see the tiny effect of the gravitational field. So ... on page 333, Griffiths has a footnote with a reference to an actual experiment that actually measured the quantum effects of a neutron “bouncing” in gravity. Oy!

(e) What value of the quantum number  $n$  is required for the 100 g tennis ball to reach a max height of 1m?

**Problem 4 : WKB Hydrogen Atom**

*adapted from Griffiths 8.14*

Your WKB derivation of the 1D SHO energy spectrum in discussion gave you the following quantization result:

$$\int_a^b p(x) dx = \left( n - \frac{1}{2} \right) \pi \hbar,$$

where  $a$  and  $b$  are the two turning points. If you look back at your work, you will see that the *form* of  $V(x)$  didn't affect the derivation of this formula at all, just the fact that it was *continuous at both turning points* and so required the Connection Formulae. Beyond its continuity, the influence of  $V(x)$  on the formula is entirely in the quantity  $p(x)$ . So, the above result is universal for continuous potentials  $V(x)$  in 1D (hence the box). We discussed in class how, in the WKB semi-classical limit,  $p(x)$  really *does* represent an approximate measure of the momentum at each location  $x$ . If we treat  $p$  as the particle's actual momentum, and calculate the integral of  $p(x) dx$  over a complete round trip from  $a$  to  $b$  to  $a =$  one period of a classical particle's motion in the well, we have derived the Wilson-Sommerfeld quantization rule from Old Quantum Theory:

$$\oint_{\text{1 period}} |p(x)| dx = \int_a^b p(x) dx + \int_b^a -p(x) dx = 2 \int_a^b p(x) dx = 2 \left( n - \frac{1}{2} \right) \pi \left( \frac{\hbar}{2\pi} \right) = \left( n - \frac{1}{2} \right) h$$

except our rule is offset by  $h/2$  from the original  $nh$ . Note that we had to play around with signs in this calculation because the " $p = -i\hbar d/dx$ " in all our calculations is actually the x-component of  $\vec{p} = -i\hbar \vec{\nabla}$ .

We can also treat the radial part of the Schrödinger equation exactly like the 1D Schrödinger equation because it has the same form ... *as long as* you remember to make the substitution  $u \equiv rR$ , don't forget the centrifugal term in the effective potential, and stay away from  $r = 0$ . (See equations 4.37, 4.38 in Griffiths.) We can thus airlift our universal WKB quantization rule for continuous 1D potentials to the radial case:

$$\int_a^b p(r) dr = \left( n - \frac{1}{2} \right) \pi \hbar.$$

(Second reminder: don't forget the centrifugal = angular-momentum term in the  $V(r)$  that goes into  $p(r)$ .) Use this WKB result to estimate the bound state energies for hydrogen. Note that you recover the Bohr energies = the results of the exact calculation when  $n \gg l$  and  $n \gg 1/2$ .