Physics 487 – Homework #11 **due Monday, Nov 18, 12:29 pm**

You may use anything from the **486/7 formula sheets** without derivation
… but do try to see how far you can get on your own on your Desert Island. ☺
You may also use **wolframalpha**.com or similar tool to evaluate your integrals
after you set the up in a form that can be **directly entered** into such tools.

Problem 1 : Precession of a Magnetic Dipole in a B-Field

Note: This is not a time-dependent perturbation theory problem, but it is a warmup for Problem 2, and provides a good review of dealing with spinors and Pauli matrices. It is also a very important calculation that we skipped over in Chapter 4. You can find help in Chapter 4, especially if you recognize this setup from classical mechanics & know exactly what phrase to look for: _ _ _ _ _ _   _ _ _ *Q* _ _ _ _ _ ☺ *)*

A spin-½ particle interacts with a magnetic field $\vec{B} = B_0\,\hat{z}$ through the interaction $H = -\mu\vec{\sigma}\cdot\vec{B}$, where μ is the particle's magnetic dipole moment and $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices. At $t = 0$, a measurement determines that the spin is pointing along the $+x$ axis. What is the probability that it will be pointing along the $-y$-axis at a later time t ? NOTE: $H = -\mu\vec{\sigma}\cdot\vec{B}$ is the entire Hamiltonian. Don't add a $p^2/2m$ term as the particle's center of mass is not moving, it is just floating in space and precessing.

Problem 2 : Magnetic Resonance *adapted from Griffiths 9.20*

A spin-½ particle has gyromagnetic ratio γ (which means that its magnetic dipole moment is $\vec{\mu} = \gamma\vec{S}$).
The particle is at rest in a static magnetic field $\vec{B} = B_0\,\hat{z}$ and precesses at the frequency $\omega_0 = \gamma\, B_0$.
(That frequency should be familiar from Problem 1! ☺). Now we turn on a small transverse radio-frequency field (universally called an "**rf field**") of the form $B_{\rm rf}\left[\hat{x}\,\cos(\omega t) - \hat{y}\,\sin(\omega t)\right]$, so that the total field is

$$\vec{B}(t) = \hat{z}\,B_0 + B_{\rm rf}\left[\hat{x}\,\cos(\omega t) - \hat{y}\,\sin(\omega t)\right]$$

(a) Construct the 2 × 2 matrix for this system's Hamiltonian, $H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{B}\cdot\vec{S}$, using as basis vectors the eigenstates of S_z.

(b) Let the spinor $\chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ represent the particle's spin state at time t. Show that $a(t)$ and $b(t)$ obey the coupled differential equations

$$\dot{a} = \frac{i}{2}\left(\Omega\, e^{i\omega t} b + \omega_0 a\right) \quad \text{and} \quad \dot{b} = \frac{i}{2}\left(\Omega\, e^{-i\omega t} a - \omega_0 b\right)$$

where $\Omega \equiv \gamma\, B_{\rm rf}$ is related to the strength of the rf field.

(c) Check that the general solution for $a(t)$ and $b(t)$, in terms of their initial values a_0 and b_0, is

$$a(t) = \left\{ a_0 \cos\left(\frac{\omega' t}{2}\right) + \frac{i}{\omega'}\left[a_0\left(\omega_0 - \omega\right) + b_0\Omega\right] \sin\left(\frac{\omega' t}{2}\right)\right\} e^{i\omega t/2}$$

$$b(t) = \left\{ b_0 \cos\left(\frac{\omega' t}{2}\right) + \frac{i}{\omega'}\left[b_0\left(\omega - \omega_0\right) + a_0\Omega\right] \sin\left(\frac{\omega' t}{2}\right)\right\} e^{-i\omega t/2} \quad \text{where } \omega' \equiv \sqrt{\left(\omega - \omega_0\right)^2 + \Omega^2}\,.$$

(d) If the particle starts out with spin up (i.e., $a_0 = 1$, $b_0 = 0$), find the probability $P(t)_{\text{up}\,\to\,\text{down}}$ of a transition to spin down as a function of time. You should get this result :

$$P(t) = \frac{\Omega^2}{(\omega - \omega_0)^2 + \Omega^2} \sin^2\left(\frac{\omega' t}{2}\right)$$

(e) The **resonance curve** shows the behaviour of the transition probability P as a function of frequency rather than as a function of time. Here we are interested in P as a function of ω = the "driving frequency" = the frequency of the perturbing rf field responsible for any transitions. If you average your $P(t)$ expression over time and normalize the result so that its maximum value is 1, you get this resonance curve (you don't have to derive it, it's simple math):

$$P(\omega) = \frac{\Omega^2}{(\omega - \omega_0)^2 + \Omega^2} .$$

Sketch this curve, noting on the sketch the frequency ω at which the maximum transition probability occurs. (That is the resonance frequency). Find the FWHM="full width at half maximum", $\Delta\omega$, of the resonance curve.

(f) Since $\omega_0 = \gamma B_0$ and ω_0 is the resonant frequency of the up $\rightarrow$ down spin-flip transition, we can determine the magnetic dipole moment of a particle by measuring ω_0. In a **nuclear magnetic resonance (NMR)** experiment, the g-factor of the proton is to be measured, where $g \equiv \gamma / (e/2m)$ as we will discuss. If we use a static field of 10,000 gauss and an rf field of amplitude 0.01 gauss, what will the resonant "NMR frequency" be? Also find the width of the resonance curve, and give both of your answers in Hz.

FYI #1: An **NMR probe** can be used in the opposite way as well. The probe contains a sample of material and coils for generating the applied rf field. If it is calibrated to a precisely known magnetic field, it can then be used to measure other magnetic fields. There were NMR probes measuring the magnetic fields in the spectrometers of my thesis experiment, for example; you tune the frequency of the applied field until resonance is achieved, then the instrument tells you the value of B_0 = the static field in which the probe finds itself.

FYI #2: The **NMR phenomenon** (resonance in the spin-flip transition of a nucleus in a magnetic field) has another *very* important application : **MRI** = Magnetic Resonance Imaging.