The 1D Simple Harmonic Oscillator ™

Hamiltonian is $H(x) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$ or $\frac{p^2}{2m} + \frac{k x^2}{2}$ with two given parameters: m and (ω or $k \equiv m\omega^2$).

The problem has two intrinsic scales:

- an energy scale $\hbar\omega$
- a distance scale $x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$ $\rightarrow$ define dimensionless-position $\xi \equiv \frac{x}{x_0}$

Hamiltonian becomes : $H(\xi) = \frac{\hbar\omega}{2}\left(\xi^2 - \frac{d^2}{d\xi^2}\right)$

Energy eigenvalues : $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

Energy eigenstates : $\psi_n(x) = \left(\frac{1}{\pi x_0^2}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x}{x_0}\right) e^{-\frac{x^2}{2x_0^2}}$ where $H_n(\xi)$ = Hermite Polynomials

Hermite Polynomials

$$H_0(\xi) = 1$$
$$H_1(\xi) = 2\xi$$
$$H_2(\xi) = 4\xi^2 - 2$$
$$H_3(\xi) = 8\xi^3 - 12\xi$$

Recursion Formula

$$H_n(\xi) = a_i \xi^i + a_{i+2}\xi^{i+2} + a_{i+4}\xi^{i+4} + \cdots$$

where $i = 0$ or 1 and

$$a_{j+2} = -a_j \frac{2(n-j)}{(j+1)(j+2)}$$

Rodrigues Formula

$$H_n(\xi) = (-1)^n e^{\xi^2} \left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

Ladder operators : $\hat{a}_\pm = \frac{m\omega x \mp ip}{\sqrt{2\hbar m\omega}} = \frac{1}{\sqrt{2}}\left(\xi \mp \frac{d}{d\xi}\right)$ $\rightarrow$ $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$, $\hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n \psi_0 \qquad H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right) \qquad [\hat{a}_-, \hat{a}_+] = 1$$