Feynman's Trick I: Differentiating Under the Integral Sign

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Throughout this course and later on in your potential physics career, you will always run across having to compute moments of exponential and Gaussian distributions. This technique that I'm about to explain is usually taught in quantum field theory for physicists and is essential in that subject. But it's simple enough to be explained here.

Rather than resorting to Wolfram Alpha or Mathematica immediately, there's an easy way to derive the moments of these integrals in general, without having to do any tedious integration.

1 Exponential Integrals

Consider the simple, innocuous integral

$$\int_0^\infty e^{-\alpha r}\,dr = \left[-\frac{1}{\alpha}e^{-\alpha r}\right]\Big|_0^\infty = \frac{1}{\alpha} \tag{1}$$

That wasn't hard to do at all. Now let's try evaluating the following,

$$\int_0^\infty re^{-\alpha r}\,dr \tag{2}$$

There are two ways to evaluate this integral, both of which are easy (at this stage) but one is slightly faster. Let's choose the method you might be inclined to use on this integral: integration by parts. If you choose to do it this way then you'll have

$$\int_0^\infty re^{-\alpha r}\,dr = \left[-\frac{r}{\alpha}e^{-\alpha r}\right]\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty e^{-\alpha r}\,dr = \frac{1}{\alpha^2} \tag{3}$$

Not bad at all. You might have had to write out $u = d(uv) - v\,du$ where $u = r$ and $dv = e^{-\alpha r}\,dr$. But what if I told you another way you could obtain this answer? One that's potentially faster?

Let's start at (1) again. Consider taking the *derivative with respect to* α on both sides

$$\frac{\partial}{\partial\alpha}\left(\int_0^\infty e^{-\alpha r}\,dr\right) = \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha}\right) = -\frac{1}{\alpha^2} \tag{4}$$

Now because of some theorem in real analysis I'd imagine, I can pull the derivative with respect to α *inside the integrand* on the left hand side of (4). Watch what happens,

$$\int_0^\infty \frac{\partial}{\partial\alpha}(e^{-\alpha r})\,dr = -\int_0^\infty r^{-\alpha r}\,dr = -\frac{1}{\alpha^2} \rightarrow \int_0^\infty re^{-\alpha r}\,dr = \frac{1}{\alpha^2}! \tag{5}$$

I'm not sure about you but that felt easier than doing an integration by parts, and we got the same answer! By differentiating under the integral sign, we could obtain the value of this integral fairly quickly.

Using this trick, we can now evaluate the following really quickly

$$\int_0^\infty r^2 e^{-\alpha r}\, dr = \int_0^\infty \left(-\frac{\partial}{\partial\alpha}(re^{-\alpha r})\right) dr = -\frac{\partial}{\partial\alpha}\left(\int_0^\infty re^{-\alpha r}\, dr\right) = -\frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^2}\right) = \frac{2}{\alpha^3} \tag{6}$$

You can probably see the pattern now:

$$\begin{aligned}
\int_0^\infty e^{-\alpha r}\, dr &= \frac{1}{\alpha} \\
\int_0^\infty re^{-\alpha r}\, dr &= -\frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^2} \\
\int_0^\infty r^2e^{-\alpha r}\, dr &= -\frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^2}\right) = \frac{2}{\alpha^3} \\
\int_0^\infty r^3e^{-\alpha r}\, dr &= -\frac{\partial}{\partial\alpha}\left(\frac{2}{\alpha^3}\right) = \frac{6}{\alpha^4}
\end{aligned} \tag{7}$$

In general, the n^{th} of the exponential distribution is given by

$$\int_0^\infty r^n e^{-\alpha r}\, dr = (-1)^n \frac{\partial^n}{\partial\alpha^n}\left(\int_0^\infty e^{-\alpha r}\, dr\right) = (-1)^n \frac{\partial^n}{\partial\alpha^n}\left(\frac{1}{\alpha}\right) = \frac{(1)(2)(3)(4)\cdots(n)}{\alpha^{n+1}} = \frac{n!}{\alpha^{n+1}} \tag{8}$$

or to put it simply,

$$\boxed{\int_0^\infty r^n e^{-\alpha r}\, dr = \frac{n!}{\alpha^{n+1}}} \tag{9}$$

For HW 5 Problem 2 b), you might find the following integrals useful. These can all be derived using Feynman's trick:

$$\boxed{\int_0^r e^{-\alpha r'}\, dr' = \frac{1}{\alpha}(1 - e^{-\alpha r}) \qquad \int_r^\infty e^{-\alpha r'}\, dr' = \frac{1}{\alpha}e^{-\alpha r}}$$

$$\boxed{\int_0^r r' e^{-\alpha r'}\, dr' = \frac{1}{\alpha^2}(1 - (1+\alpha r)e^{-\alpha r}) \qquad \int_r^\infty r' e^{-\alpha r'}\, dr' = \frac{1}{\alpha^2}(1+\alpha r)e^{-\alpha r}}$$

$$\boxed{\int_0^r r'^2 e^{-\alpha r'}\, dr' = \frac{1}{\alpha^3}(2 - 2(1+\alpha r)e^{-\alpha r} - \alpha^2 r^2 e^{-\alpha r}) \qquad \int_r^\infty r'^2 e^{-\alpha r'}\, dr' = \frac{1}{\alpha^3}(2(1+\alpha r) + \alpha^2 r^2)e^{-\alpha r}} \tag{10}$$

2 Gaussian Integrals

If you recall from Physics 486, we showed that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}} \tag{11}$$

Because the $e^{-\alpha x^2}$ is an even function, it's no surprise that

$$\int_{-\infty}^{\infty} x^{2k+1} e^{-\alpha x^2}\,dx = 0 \tag{12}$$

Now starting from (11) let's use Feynman's trick to evaluate

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx \tag{13}$$

This doesn't take too long.

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx &= \int_{-\infty}^{\infty} \left(-\frac{\partial}{\partial\alpha}(e^{-\alpha x^2})\right) dx = -\frac{\partial}{\partial\alpha}\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx\right) = -\frac{\partial}{\partial\alpha}\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx\right) \\ &= -\frac{\partial}{\partial\alpha}\left(\sqrt{\frac{\pi}{\alpha}}\right) = \frac{\sqrt{\pi}}{2\alpha^{\frac{3}{2}}}\end{aligned} \tag{14}$$

With the exponential integrals it may have not been obvious but this method definitely beats using integration by parts (try it yourself and see how much longer it takes). You can establish a similar pattern as we did in (7):

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx &= \sqrt{\frac{\pi}{\alpha}} \\ \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx &= -\frac{\partial}{\partial\alpha}\left(\sqrt{\frac{\pi}{\alpha}}\right) = \frac{\sqrt{\pi}}{2\alpha^{\frac{3}{2}}} \\ \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2}\,dx &= -\frac{\partial}{\partial\alpha}\left(\frac{\sqrt{\pi}}{2\alpha^{\frac{3}{2}}}\right) = \frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}} \\ \int_{-\infty}^{\infty} x^6 e^{-\alpha x^2}\,dx &= -\frac{\partial}{\partial\alpha}\left(\frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}}\right) = \frac{15\sqrt{\pi}}{8\alpha^{\frac{7}{2}}}\end{aligned} \tag{15}$$

Averages (or moments more specifically) are a cinch.

$$\langle x^2 \rangle = \frac{\int\limits_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\, dx}{\int\limits_{-\infty}^{\infty} e^{-\alpha x^2}\, dx} = \frac{1}{2\alpha}$$

$$\langle x^4 \rangle = \frac{\int\limits_{-\infty}^{\infty} x^4 e^{-\alpha x^2}\, dx}{\int\limits_{-\infty}^{\infty} e^{-\alpha x^2}\, dx} = \frac{3}{4\alpha^2} \tag{16}$$

$$\langle x^6 \rangle = \frac{\int\limits_{-\infty}^{\infty} x^6 e^{-\alpha x^2}\, dx}{\int\limits_{-\infty}^{\infty} e^{-\alpha x^2}\, dx} = \frac{15}{8\alpha^3}$$

In general, we know from (12) that $\langle x^{2k+1} \rangle = 0$. From (16), following the pattern, we can show that

$$\boxed{\langle x^{2k} \rangle = \frac{\int\limits_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2}\, dx}{\int\limits_{-\infty}^{\infty} e^{-\alpha x^2}\, dx} = \sqrt{\frac{\alpha}{\pi}} \left((1)(3)(5)\cdots(2k+1) \frac{\sqrt{\pi}}{\alpha^{\frac{2k+1}{2}}} \right) \equiv \frac{(2k+1)!!}{\alpha^k}} \tag{17}$$