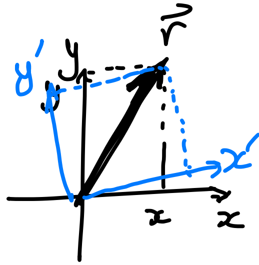
list $\vec{t} = \begin{pmatrix} \text{temp}_1 \\ \text{temp}_2 \\ \text{temp}_3 \end{pmatrix}$ ... cf. $\boxed{\text{vector}}$ $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$

difference = TRANFORMATION PROPERTIES



- A 3D vector ≡ any list of 3 numbers that <u>transforms like</u> $\vec{r}$ under <u>rotations</u>

ex. $\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$, $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{\text{VECTOR}}{\text{scalar}} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

all rotate like

$\vec{v}' = \mathbb{R}\,\vec{v}$

- A 4-vector ≡ any list of 4 numbers that <u>transforms like</u> $x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ under <u>rotations</u> & <u>boosts</u> $\rightarrow x'^\mu = \Lambda x^\mu$

e.g. $p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$

---

number = 42 .... cf. $\boxed{\text{scalar}}$ = $|\vec{r}|^2 = \vec{r}\cdot\vec{r} = r_i r_i$

= LENGTH

- scalar ≡ number that does <u>not change</u> under <u>rotations</u>

e.g. $|\vec{p}| = \sqrt{\vec{p}\cdot\vec{p}}$, etc...

- 4-scalar ≡ number that doesn't <u>not change</u> under <u>rotations</u> OR <u>boosts</u>

e.g. $d\tau$ or $\Delta\tau$

---

table = or MATRIX

1st initial: HOME STATE	A	B	C	...
1L	3	2	6	
OH	0	1	0	
1A	0	0	2	
⋮				

cf. **Tensor**

I_{ij} inertia tensor

$$I_{ij} = \int dm \left(\delta_{ij} \underset{\uparrow \text{SCALAR}}{r^2} - r_i r_j \right)$$

Tensor = matrix that transforms under <u>rotations</u>

like $T_{ij} \equiv r_i r_j$ ... and how do we do that? ...

"Tensor of rank n" $\equiv$

scalar	if	$n=0$	λ
vector	if	$n=1$	r_i
tensor	if	$n=2$	T_{ij}
"rank 3" tensor	if	$n=3$	M_{ijk}
⋮	if	⋮	⋮

↑ generalized use of word tensor

Transformation of a tensor

Replace "rotation" with "change of basis"

vector $\vec{V}$ START in some basis: "old" $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

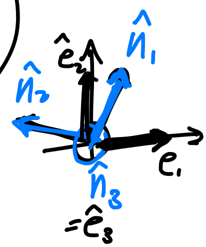
TRANSFORM to a "new basis" $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$

This new basis ≈ always defined in the old basis.

What is the new basis $\{\hat{n}_i\}$ expressed in the new basis?

IN TERMS OF $\Longrightarrow$ OLD: BASIS

$$\hat{n}_1 = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}, \quad \hat{n}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



IN TERMS OF $\Longrightarrow$ NEW BASIS

$$\hat{n}'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{n}'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{n}'_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Define transformation matrix $\mathbb{R}$ for vectors:

↳ changes from old → new basis

$$\boxed{\underbrace{\vec{V}'}_{\text{in NEW basis}} = \mathbb{R} \underbrace{\vec{V}}_{\text{in OLD basis}}}$$

Build $\mathbb{R}$ ..... no, build $\mathbb{R}^{-1}$:

we know that $\mathbb{R}^{-1} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{vector in NEW basis}} = \underbrace{\hat{n}_1}_{\text{vector in OLD basis}} =$ basis in which you know $\hat{n}_1$!! ☺ $= \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$

$$\mathbb{R}^{-1}\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{in New basis}} = \underbrace{\hat{n}_2}_{\text{in old basis}} = \begin{pmatrix}-4/5\\3/5\\0\end{pmatrix}, \quad \text{etc}\ldots$$

$\forall$ matrix $\mathbb{M}$, $\mathbb{M}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} = $ 1st column of $\mathbb{M}$, $\mathbb{M}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} = $ 2nd column of $\mathbb{M}$, ...

$$\therefore \boxed{\mathbb{R}^{-1} = \begin{pmatrix} | & | & | \\ \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ | & | & | \end{pmatrix}}$$

INVARIANTS

How to transform a tensor? understand $\underline{\text{scalar combinations}}$

$\vec{u}\cdot\vec{v} = $ scalar $\overset{\text{check}}{\Rightarrow}$ should have $\vec{u}\cdot\vec{v} = \vec{u}'\cdot\vec{v}'$ ... true?

$$\vec{u}\cdot\vec{v} = \underbrace{\vec{u}^{T}\,\vec{v}}_{\text{in matrix notation}} = (\mathbb{R}\vec{u}')^{T}(\mathbb{R}\vec{v}') = \vec{u}'^{T}\underbrace{\mathbb{R}^{-1^{T}}(\mathbb{R}^{-1}}\vec{v}') \overset{?}{=} \vec{u}'^{T}\vec{v}'$$

YES if $\mathbb{R}^{-1^{T}}\mathbb{R}^{-1} = \mathbb{1}$

unit matrix = identity matrix

$$\mathbb{R}^{-1^{T}}\mathbb{R}^{-1} = \begin{pmatrix} — \hat{n}_1 — \\ — \hat{n}_2 — \\ — \hat{n}_3 — \end{pmatrix}\begin{pmatrix} | & | & | \\ \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} \hat{n}_1\cdot\hat{n}_1 & \hat{n}_1\cdot\hat{n}_2 & \cdot \\ \hat{n}_2\cdot\hat{n}_1 & \hat{n}_2\cdot n_2 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} \text{ as long as } \begin{array}{l} \hat{n}_i \cdot \hat{n}_i = 1 \\ \hat{n}_i \cdot \hat{n}_j = 0 \text{ if } i \neq j \end{array}$$

i.e. $\hat{n}_i \cdot \hat{n}_j = \delta_{ij}$

<u>assume this</u> ← basis is <u>ORTHONORMAL</u>

Then $(R^{-1})^T R^{-1} = \mathbb{1}$ $\therefore$ $(R^{-1})^T = R$ i.e. R is orthonormal matrix

- $\therefore$ $\vec{u} \cdot \vec{v} = \vec{u}' \cdot \vec{v}' =$ SCALAR
- matrix notation: $\vec{u}^T \vec{v} = \vec{u}'^T \vec{v}'$ → $\left(-\vec{u}- \right) \begin{pmatrix} | \\ \vec{v} \\ | \end{pmatrix} =$ scalar
- index notation: $u_i v_i = u_j' v_j' = \overset{\text{SCALAR}}{\lambda}$ with implicit sums $\sum_{i=1}^{3}$ & $\sum_{j=1}^{3}$

CONTRACTED INDEX → ie. it disappears → zero-index result = a scalar

produced by dot product a.k.a. <u>inner product</u>: inner product of * two rank-1 tensors = rank-0 tensor

COMPARE <u>outer product</u> of two vectors:

- $\vec{u} \otimes \vec{v} =$ TENSOR
- $\vec{u}\,\vec{v}^T$ in matrix notation → $\begin{pmatrix} | \\ \vec{u} \\ | \end{pmatrix} \left(-\vec{v}- \right) = \overset{\text{tensor}}{\begin{pmatrix} u_1v_1 & u_1v_2 & \cdot \\ u_2v_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$
- $u_i v_j$ in index notation

"DOUBLED" INDEX → two-index result = a tensor : outer product of * two rank-1 tensors = rank-2 tensor

Now we have the tools to build a scalar from a tensor:

λ $\qquad$ $\mathbb{M}$

① $\text{SCALAR} = \vec{v}^{\,T} \mathbb{M} \vec{u} = (-\vec{v}-)\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} | \\ \vec{u} \\ | \end{pmatrix} = (-\vec{v}-)\begin{pmatrix} | \\ | \\ | \end{pmatrix}$

$= \text{SCALAR}$ ($\emptyset$ indices)

matrix not$^{\underline{n}}$

$= \vec{v}'^{\,T} \underline{\underline{\mathbb{M}'}} \vec{u}'$ invariant!

$= (\mathbb{R}^{-1}\vec{v}')^T \mathbb{M} (\mathbb{R}^{-1}\vec{u}')$

SAME

$= \vec{v}'^{\,T} \underbrace{\mathbb{R}^{-1\,T} \mathbb{M}\, \mathbb{R}^{-1}} \vec{u}'$

$$\boxed{\mathbb{M}' = \mathbb{R}^{-1\,T} \mathbb{M}\, \mathbb{R}^{-1}}$$

② another derivation: $\mathbb{M} \equiv \vec{u} \otimes \vec{v} = \vec{u}\,\vec{v}^{\,T}$

$\mathbb{M}' = \vec{u}' \otimes \vec{v}' = \vec{u}'\,\vec{v}'^{\,T} = (\mathbb{R}\vec{u})(\mathbb{R}\vec{v})^T = \mathbb{R}\,\underbrace{\vec{u}\,\vec{v}^{\,T}}\,\mathbb{R}^T$

$$\boxed{\mathbb{M}' = \mathbb{R}\,\mathbb{M}\,\mathbb{R}^T}$$

$\mathbb{R} \to (\mathbb{R}^{-1})^T$ $\qquad$ $\mathbb{R}^T \to \mathbb{R}^{-1}$

$\because \mathbb{R}$ = orthonormal

$\therefore$ same as xform$^{\underline{n}}$ rule in box

$$\begin{pmatrix} -\hat{e}_1- \\ -\hat{e}_2- \\ -\hat{e}_3- \end{pmatrix}\begin{pmatrix} | & | & | \\ \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ | & | & | \end{pmatrix} = \delta_{ij} = \mathbb{1}$$

$\mathbb{M}$

$\mathbb{R}^{-1} = \mathbb{R}^T$ $\qquad$ $\mathbb{R}$

NOTE ①: When vectors, scalars, tensors are COMPLEX, then

- orthogonal $\mathbb{R}$ matrix → unitary $\mathbb{R}$ matrix
- Transposes replaced by → Adjoint $^{\dagger} = T^{*}$
  i.e. add a complex conjugate to the T operation

COMPLEX

$$\mathbb{R}^{-1} = \begin{pmatrix} | & | & | \\ \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ | & | & | \end{pmatrix} \cdots \{\hat{n}_i\} \text{ orthonormal}$$

means $\langle \hat{n}_i | \hat{n}_j \rangle = \hat{n}_i^{T*} \hat{n}_j = \delta_{ij}$

$$(\mathbb{R}^{-1})^{T*} \mathbb{R}^{-1} = \begin{pmatrix} — & \hat{n}_1^* & — \\ — & \hat{n}_2^* & — \\ — & \hat{n}_3^* & — \end{pmatrix} \begin{pmatrix} | & | & | \\ \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ | & | & | \end{pmatrix} = \delta_{ij} = \mathbb{1}$$

NOTE ②: Re-check Inertia tensor

$$I_{ij} = \int dm(\delta_{ij} r^2 - \underbrace{r_i r_j}_{\vec{r}\vec{r}^T = \vec{r} \otimes \vec{r}})$$

$\cdots$ IS it a tensor?

$\vec{r}\vec{r}^T = \vec{r} \otimes \vec{r}$ = OUTER PRODUCT of 2 vectors ✓

really? TENSOR

SCALAR

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} \cdots$$

index notation $\quad$ matrix

transform: $\mathbb{1}' = \mathbb{R}^{-1T} \mathbb{1} \mathbb{R}^{-1}$

UNIT TENSOR

$\mathbb{1} = \delta_{ij}$

is unchanged under rotation

$= \mathbb{R}^{-1T}\,\mathbb{R}^{-1}$

$= \mathbb{R}\,\mathbb{R}^{-1} = \mathbb{1} = \delta_{ij}$

in both frames

$\therefore\ \mathbb{I}$ YES $= \overset{\delta_{ij}}{\text{tensor}} \cdot \overset{r^2}{\text{scalar}} - \overset{r_i r_j}{\text{tensor}}$

$= \text{tensor}$ ✓