

Physics 487 – Homework #4

due Friday Sep 25 @ midnight: 1 file → my.physics Course Uploads

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: our master rule for homework and exams is **NO WORK = NO POINTS**. However, unless otherwise specified, you may use any relation from the **formula sheets** on our website without proof. **GUIDANCE:** Write enough so that it will make sense to YOU in five years, i.e. so that it provides you with useful future notes. Finally please write your **NAME** and **DISCUSSION SECTION** on your solutions. ☺

You may always use anything from the **formula sheets** on our web site without derivation.
For this homework, you may also use wolframalpha.com , wolframcloud.com,
or any similar tool to evaluate your integrals
after you set the up in a form that can be **directly entered** into such tools.

Reading for week 5 : Griffiths §6.1 and 6.2

Problem 1 : Full Wavefunction = $\psi(\text{Space}) \times \chi(\text{Spin})$

adapted from Griffiths 4.55

The electron in a hydrogen atom is in the following state :

$$\psi = R_{21} \left(\sqrt{\frac{1}{3}} Y_1^0 \chi_+ + \sqrt{\frac{2}{3}} Y_1^1 \chi_- \right)$$

where the basis states for the spinors are, as usual,

$$\chi_+ \equiv \chi_{(m_s=+1/2)} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi_- \equiv \chi_{(m_s=-1/2)} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

- (a) If you measured L^2 , what values might you get, and what is the probability of each?
- (b) same question for L_z (c) same question for S^2 (d) same question for S_z
- (e) same question for J^2 where $\vec{J} \equiv \vec{L} + \vec{S}$ (f) same question for J_z
- (g) If you measured the position of the electron, what is the probability density for finding it at (r, θ, ϕ) ?
- (h) If you measured both the z component of the spin *and* the distance from the origin (note that these are compatible¹ observables), what is the probability density for finding the electron with spin up at radius r ?

Problem 2 : Helium Ground State Energy

(b) adapted from Griffiths 5.11

(a) The helium atom consists of two electrons bound by a doubly-charged nucleus. The most crude approximation you can make to its Hamiltonian is the sum of two hydrogen-like Hamiltonians, i.e. one for each of the two electrons, treating them as if they are interacting with just the Z (nuclear charge) = 2 nucleus and not at all with each other (also no spin-orbit interaction or other corrections to the Bohr model). Making use of the hydrogenic energy eigenstates and eigenvalues,

¹ Jargon check: Two observables A and B are **compatible** if they **commute** ... which means that they share a **common set of eigenstates** ... which means that if you measure A, thereby placing the system in some eigenstate $|a\rangle$ of A, the state $|a\rangle$ will also be an eigenstate of B — let's call it $|a; b\rangle$ — so you can go ahead and measure B and obtain the value b without further disturbing the state! You won't change the value of A when you measure B, hence A & B are "compatible observables". (There is a small subtlety involving degenerate states, but it's not much of an issue: " b " can be a set of degenerate eigenvalues of B, that's all; you still won't disturb the eigenvalue a when measuring B.)

$$\psi_{nlm}(\vec{r}) = R_{nl}(r) Y_{lm}(\theta, \phi) \quad \text{and} \quad E_n = -\frac{(Z\alpha)^2}{2n^2} (m_e c^2),$$

from our 486 formula sheet, calculate the energy of the helium ground state in the approximation that Helium \approx two non-interacting electrons sitting in the field of a $Z=2$ nucleus. Express your result in eV.

(b) The measured energy of the helium ground state is -79 eV, so the non-interacting-particle approximation is not very accurate (almost 40% tighter binding than -79 eV). Almost all of the discrepancy with experiment can be accounted for by adding the **electron-electron repulsion term**, V_{ee} , to the Hamiltonian. Calculate the expectation value $\langle V_{ee} \rangle$ of this interaction for the helium ground state. After you apply $\langle V_{ee} \rangle$ correction, you should be about 5% away from the experimental ground state energy instead of 40% \rightarrow as we said during our discussion of the Hund rules, V_{ee} is the dominant correction to the Bohr model for multi-electron atoms.

► INTEGRATION GUIDANCE: Calculating $\langle V_{ee} \rangle$ requires calculating $\langle 1/|\vec{r}_1 - \vec{r}_2| \rangle$. See Griffiths problem 5.11(a) for assistance on how to set up this multi-dimensional integral. Also see Saavanth Velury's excellent writeup on Feynman's Trick for integration!

Problem 3 : The Fourth Row

The electronic configurations of the first four elements in the fourth row of the periodic table are in the box :

<div style="border: 1px solid black; padding: 5px; display: inline-block;"> <p>Z=19 K [Ar] 4s¹</p> <p>Z=20 Ca [Ar] 4s²</p> <p>Z=21 Sc [Ar] 4s² 3d¹</p> <p>Z=22 Ti [Ar] 4s² 3d²</p> </div>																			
1 IA 11A	2 IIA 2A													13 IIIA 3A	14 IVA 4A	15 VA 5A	16 VIA 6A	17 VIIA 7A	18 VIIIA 8A
1 H Hydrogen 1.008																	2 He Helium 4.003		
3 Li Lithium 6.941	4 Be Beryllium 9.012												5 B Boron 10.811	6 C Carbon 12.011	7 N Nitrogen 14.007	8 O Oxygen 15.999	9 F Fluorine 18.998	10 Ne Neon 20.180	
11 Na Sodium 22.990	12 Mg Magnesium 24.305	3 IIIB 3B	4 IVB 4B	5 VB 5B	6 VIB 6B	7 VIIB 7B	8 VIII 8	9 VIII 8	10 VIII 8	11 IB 1B	12 IIB 2B	13 Al Aluminum 26.982	14 Si Silicon 28.086	15 P Phosphorus 30.974	16 S Sulfur 32.066	17 Cl Chlorine 35.453	18 Ar Argon 39.948		
19 K Potassium 39.098	20 Ca Calcium 40.078	21 Sc Scandium 44.956	22 Ti Titanium 47.88	23 V Vanadium 50.942	24 Cr Chromium 51.996	25 Mn Manganese 54.938	26 Fe Iron 55.933	27 Co Cobalt 58.933	28 Ni Nickel 58.693	29 Cu Copper 63.546	30 Zn Zinc 65.39	31 Ga Gallium 69.732	32 Ge Germanium 72.61	33 As Arsenic 74.922	34 Se Selenium 78.09	35 Br Bromine 79.904	36 Kr Krypton 84.80		

► Recall from chemistry: The word **valence** comes from the Latin word for “power”. A **valence electron** is “an electron available for chemical bonding”, i.e. an electron in a partly-filled shell. The **noble gases** He, Ne, Ar, ... have *no* valence electrons, and so are tightly-bound systems that hardly interact with other atoms. To simplify electronic configurations, it is common to lump together all the chemically-inactive electrons from the nearest noble gas into a single symbol, as you see in the box above. Expanded out, the e^- configuration for potassium is $1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$; it's much simpler to absorb all those filled shells into one symbol: $[\text{Ar}] 4s^1$.

(a) Consider potassium (K). Its ground state is $^2S_{1/2}$ = the “doublet-S- $1/2$ ”. Explain briefly why this is the only possible ground state given that the ground state electron configuration is $\text{K} = [\text{Ar}] 4s^1$.

(b) As you see from the box, scandium (Sc) has only one valence electron. First, list all possible $^{2S+1}L_J$ states given that the valence electron is in the d -shell. Then decide which of your states has the lowest energy and explain your decision qualitatively (no calculation needed).

(c) Titanium (Ti) has two valence electrons, as you can see from its electron configuration in the box above. As it happens, the ground state of Ti has total orbital angular momentum quantum number $L = 3$. Let's align a ground-state Ti atom so that its M_L quantum number is -1 . *Ignoring spin entirely*, express this state, $|3, -1\rangle_{LM}$, as a superposition of states $|m_1 m_2\rangle$ using Clebsch-Gordan tables.

(d) The ground state of Ti is the “triplet-F-2” state, 3F_2 . Explain the triplet part: why does the system reach its lowest energy when its total spin is 1? Please explain more deeply than simply quoting one of the Hund rules, you need to explain the *reason* behind that Hund rule.

(e) What is your favourite element?