Useful Formulas

Quantum Mechanics Fundamentals

- Canonical Commutation Relations: $[\hat{x}, \hat{p}] = i\hbar$
- Position-basis representations: $\hat{x} f(x) = x f(x), \quad \hat{p} f(x) = -i\hbar \frac{df}{dx}$
- Schrödinger Equation: $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$

Perturbation Theory

Time-independent perturbation theory

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \lambda \hat{H}' \\
|\psi_n\rangle &= \left|\psi_n^{(0)}\right\rangle + \lambda \left|\psi_n^{(1)}\right\rangle + \ldots, \\
E_n &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots
\end{aligned}$$

Non-degenerate case

$$E_n^(1) = \left\langle \psi_n^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle, \quad \left|\psi_n^{(1)}\right\rangle = -\sum_{m \neq n} \left|\psi_m^{(0)}\right\rangle \frac{\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle}{E_m^{(0)} - E_n^{(0)}}, \quad E_n^{(2)} = -\sum_{m \neq n} \frac{\left|\left\langle \psi_m^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle\right|^2}{E_m^{(0)} - E_n^{(0)}}$$

Degenerate case

Let $\left|\psi_{ni}^{(0)}\right\rangle, i = 1, 2, \ldots d$ be the eigenstates of $\hat{H}_0$ that span the degenerate subspace. Find $\left|\psi_{ni}^{(0),\text{good}}\right\rangle$, the eigenvectors of $\hat{H}'$ restricted to the degenerate subspace. Then $E_{ni}^{(1)} = \left\langle \psi_{ni}^{(0),\text{good}} \middle| \hat{H}' \middle| \psi_{ni}^{(0),\text{good}} \right\rangle$.

Heisenberg and Schrödinger Pictures

- For time-independent Hamiltonians, $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$
- Schrödinger picture: $|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$
- Heisenberg picture: $\hat{O}_H(t) = \hat{U}^\dagger(t) \hat{O}_H(0) \hat{U}(t)$
- Heisenberg equations of motion: $\frac{\partial \hat{O}_H}{\partial t} = \frac{i}{\hbar} \left[\hat{H}, \hat{O}_H(t)\right]$

Simple Harmonic Oscillator

$$
\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \\
\hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} + i\hat{p}\right), \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} - i\hat{p}\right), \\
\left|n\right\rangle &= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle, \\
\hat{a}\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle, \\
\hat{a}^\dagger\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle
\end{aligned}
$$

Pauli Matrices

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$