# Fall 2023

1. **Review from 486**

- Canonical commutation relations:
$$[\hat{x}, \hat{p}] = i\hbar \tag{1}$$
So that $\hat{p} = -i\hbar\frac{d}{dx}$ for wavefunctions in the positions basis $\psi(x) = \langle x|\,\psi\rangle$. Similarly $\hat{x}\psi(x) = x\psi(x)$.
- Schrödiner equation:
$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle \tag{2}$$
- Inner product for wavefunctions:
$$\langle\psi|\,\varphi\rangle = \int dx\psi(x)^{\star}\varphi(x) \tag{3}$$
- Energy eigenstates:
$$\hat{H}|E_n\rangle = E_n|E_n\rangle \qquad \langle E_n|\,E_m\rangle = 0\ E_n \neq E_m \tag{4}$$
- The average of an observable: $\langle A\rangle = \langle\psi|\,A\,|\psi\rangle$ and fluctuations $\sigma_A^2 = \langle A^2\rangle - \langle A\rangle^2$.
- Useful operator relation $[A, BC] = [A, B]C + B[A, C]$ where $[A, B] = AB - BA$
- Simple Harmonic oscillator:
$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(\hat{N} + 1/2) \qquad \hat{N} = a^{\dagger}a \tag{5}$$
for creation and annihilation operators satisfying:
$$a = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}) \qquad [a, a^{\dagger}] = 1 \tag{6}$$
such that:
$$\hat{N}|n\rangle = n|n\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{7}$$
- Abstract angular momentum: $J_i$ where $i = x, y, z$ satisfies:
$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \qquad [J^2, J_i] = 0 \qquad J^2 = J_x^2 + J_y^2 + J_z^2 \tag{8}$$
with eigenfunctions:
$$J^2\left|jm_j\right\rangle = \hbar^2 j(j+1)\left|jm\right\rangle \qquad J_z\left|jm_j\right\rangle = \hbar m_j\left|jm_j\right\rangle \tag{9}$$
with possible values $j = 0, 1/2, 1, 3/2\ldots$ and $m_j = -j, -j+1, \ldots j$.

- Pauli matrices:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{10}$$

2. **Symmetries**

- Unitary transformation $U^\dagger U = UU^\dagger = 1$. Continuous symmetries given by a generator $U(a) = \exp(-i\hat{G}a/\hbar)$ where $\hat{G}$ is Hermitian.
- Translations:

$$T(a) = \exp(-i\hat{p}a/\hbar) \qquad \langle x| T(a) |\psi\rangle = \psi(x-a) \qquad T(a)^\dagger \hat{x} T(a) = \hat{x} + a \tag{11}$$

- Rotations: $U(\chi,\hat{n}) = \exp(-i\chi\vec{L}\cdot\hat{n}/\hbar)$
- Symmetries: $U^\dagger H U = H$ or $[H,\hat{G}] = 0$
- Time evolution (Schrodinger picture):

$$U(t) = \exp(-it\hat{H}/\hbar) \qquad |\psi_S(t)\rangle = U(t)|\psi(0)\rangle \tag{12}$$

- Heisenberg picture:

$$\mathcal{O}_H(t) = U(t)^\dagger \mathcal{O}(0) U(t) \tag{13}$$

3. **Perturbation Theory (Time independent)**

$$H = H_0 + \lambda V \tag{14}$$

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle \tag{15}$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots \tag{16}$$

- Non-degenerate case:

$$E_n^{(1)} = \langle\psi_n^{(0)}| V |\psi_n^{(0)}\rangle \tag{17}$$

$$|\psi_n^{(1)}\rangle = -\sum_{k\neq n} |\psi_k^{(0)}\rangle \frac{\langle\psi_k^{(0)}| V |\psi_k^{(0)}\rangle}{E_k^{(0)} - E_n^{(0)}} \tag{18}$$

$$E_n^{(2)} = \langle\psi_n^{(0)}| V |\psi_n^{(1)}\rangle \tag{19}$$

- Degenerate case: for fixed $n$ work in the degeneate subspace spanned by $|\psi_{n,i}^{(0)}\rangle$ for $i = 1,\ldots N_n$ where $N_n$ is the degenerate of the $E = E_n^{(0)}$ subspace. Find the matrix:

$$\mathbb{W}_{ij} = \langle\psi_{n,i}^{(0)}| V |\psi_{n,j}^{(0)}\rangle \tag{20}$$

Then the eigenvalues $\delta E_k$ give the corrections $E_{n,k}^{(1)} = \delta E_k$ and the “good” eigenststates are $|\widetilde{\psi}_{n,k}^{(0)}\rangle = \sum_i (\alpha_k)_i |\psi_{n,i}^{(0)}\rangle$ where $\vec{\alpha}_k$ are the normalized eigenvectors of $\mathbb{W}$ with eigenvalue $\delta E_k$.