Useful Formulas

Quantum Mechanics Fundamentals

- Canonical Commutation Relations: $[\hat{x}, \hat{p}] = i\hbar$
- Position-basis representations: $\hat{x} f(x) = x f(x), \quad \hat{p} f(x) = -i\hbar \frac{df}{dx}$
- Schrödinger Equation: $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$
- Euler's formula: $e^{ix} = \cos x + i \sin x$

Perturbation Theory

Time-independent perturbation theory

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \lambda \hat{H}' \\ |\psi_n\rangle &= \left|\psi_n^{(0)}\right\rangle + \lambda \left|\psi_n^{(1)}\right\rangle + \ldots, \\ E_n &= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots\end{aligned}$$

Non-degenerate case

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \right| \hat{H}' \left| \psi_n^{(0)} \right\rangle, \quad \left|\psi_n^{(1)}\right\rangle = -\sum_{m \neq n} \left|\psi_m^{(0)}\right\rangle \frac{\left\langle \psi_m^{(0)} \right| \hat{H}' \left| \psi_n^{(0)} \right\rangle}{E_m^{(0)} - E_n^{(0)}}, \quad E_n^{(2)} = -\sum_{m \neq n} \frac{\left| \left\langle \psi_m^{(0)} \right| \hat{H}' \left| \psi_n^{(0)} \right\rangle \right|^2}{E_m^{(0)} - E_n^{(0)}}$$

Degenerate case

Let $\left|\psi_{ni}^{(0)}\right\rangle, i = 1, 2, \ldots d$ be the eigenstates of $\hat{H}_0$ that span the degenerate subspace. Find $\left|\psi_{nj}^{(0),\text{good}}\right\rangle, j = 1, 2, \ldots d$, the eigenvectors of $\hat{H}'$ restricted to the degenerate subspace. Then $E_{nj}^{(1)} = \left\langle \psi_{nj}^{(0),\text{good}} \right| \hat{H}' \left| \psi_{ni}^{(0),\text{good}} \right\rangle$.

Heisenberg and Schrödinger Pictures

- For time-independent Hamiltonians, $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$
- Schrödinger picture: $|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$
- Heisenberg picture: $\hat{O}_H(t) = \hat{U}^\dagger(t) \hat{O}_H(0) \hat{U}(t)$
- Heisenberg equations of motion: $\frac{\partial \hat{O}_H}{\partial t} = \frac{i}{\hbar} \left[\hat{H}, \hat{O}_H(t)\right]$

Simple Harmonic Oscillator

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}), \\ \hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega \hat{x} + i\hat{p}\right), \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega \hat{x} - i\hat{p}\right), \\ |n\rangle &= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle, \\ \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle, \\ \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle\end{aligned}$$

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Variational Principle

Let E_0 be the ground state energy of H. Then

$$R_H(|\psi\rangle) = \frac{\langle\psi| H |\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

for ANY $|\psi\rangle$.

Time-dependent Hamiltonians

$H = H_0 + H'(t)$ with $E_n, |\psi_n\rangle$ energies and eigenstates of H_0:

- $$\begin{aligned} |\psi(t)\rangle &= \sum_n c_n(t) e^{-iE_n t/\hbar} |\psi_n\rangle \\ i\hbar\dot{c}_n &= \sum_m H'_{nm}(t) e^{i\omega_{nm}t} c_m \\ H'_{nm}(t) &= \langle\psi_n| H'(t) |\psi_m\rangle, \\ \hbar\omega_{nm} &= E_n - E_m \end{aligned}$$
- If $|\psi(t=0)\rangle = |\psi_n\rangle$, then $P_{n\to m}(t) = |\langle\psi_m|\psi(t)\rangle|^2 = |c_m(t)|^2$

Angular Momentum

- Commutation relations: $[J_i, J_j] = i\hbar\sum_k \epsilon_{ijk} J_k$, where i, j, k are all one of x, y, z
- Eigenstates: $|jm\rangle$, $J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle$, $J_z |jm\rangle = \hbar m |jm\rangle$. j is integer or half-integer and $m \in \{-j, -j+1, \ldots, j-1, j\}$.
- Adding two angular momenta $\vec{J} = \vec{J}_1 + \vec{J}_2$:

  $$\begin{aligned} J^2 &= J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 \\ \left[J_1^2, \vec{J}_1\right] &= \left[J_1^2, \vec{J}_2\right] = \left[J_2^2, \vec{J}_1\right] = \left[J_2^2, \vec{J}_2\right] = 0 \end{aligned}$$

  Complete set of commuting observables for total angular momentum: J_1^2, J_2^2, J^2, J_z. Clebsch-Gordan coefficients give

  $$\begin{aligned} |j_1 j_2 j m_j\rangle &= \sum_{m_1, m_2} C^{jm_j}_{j_1 j_2 m_1 m_2} |j_1 j_2 m_1 m_2\rangle \\ C^{jm_j}_{j_1 j_2 m_1 m_2} &= \langle j_1 j_2 j m_j | j_1 j_2 m_1 m_2\rangle \end{aligned}$$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1