

Fall 2024

1. Review from 486

- Canonical commutation relations:

$$[\hat{x}, \hat{p}] = i\hbar \quad (1)$$

So that $\hat{p} = -i\hbar \frac{d}{dx}$ for wavefunctions in the positions basis $\psi(x) = \langle x | \psi \rangle$. Similarly $\hat{x}\psi(x) = x\psi(x)$.

- Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \quad (2)$$

- Inner product for wavefunctions:

$$\langle \psi | \varphi \rangle = \int dx \psi(x)^* \varphi(x) \quad (3)$$

- Energy eigenstates:

$$\hat{H} |E_n\rangle = E_n |E_n\rangle \quad \langle E_n | E_m \rangle = \delta_{nm} \quad E_n \neq E_m \quad (4)$$

- The average of an observable: $\langle A \rangle = \langle \psi | A | \psi \rangle$ and fluctuations $\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2$.
- Useful operator relation $[A, BC] = [A, B]C + B[A, C]$ where $[A, B] = AB - BA$
- Simple Harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega(\hat{N} + 1/2) \quad \hat{N} = a^\dagger a \quad (5)$$

for creation and annihilation operators satisfying:

$$a = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega\hat{x} + i\hat{p}) \quad [a, a^\dagger] = 1 \quad (6)$$

such that:

$$\hat{N} |n\rangle = n |n\rangle \quad a |n\rangle = \sqrt{n} |n-1\rangle \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad (7)$$

- Abstract angular momentum: J_i where $i = x, y, z$ satisfies:

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k \quad [J^2, J_i] = 0 \quad J^2 = J_x^2 + J_y^2 + J_z^2 \quad (8)$$

with eigenfunctions:

$$J^2 |jm_j\rangle = \hbar^2 j(j+1) |jm_j\rangle \quad J_z |jm_j\rangle = \hbar m_j |jm_j\rangle \quad (9)$$

with possible values $j = 0, 1/2, 1, 3/2, \dots$ and $m_j = -j, -j+1, \dots, j$.

- Pauli matrices:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (10)$$

2. Symmetries

- Unitary transformation $U^\dagger U = U U^\dagger = 1$. Continuous symmetries given by a generator $U(a) = \exp(-i\hat{G}a/\hbar)$ where \hat{G} is Hermitian.
- Translations:

$$T(a) = \exp(-i\hat{p}a/\hbar) \quad \langle x|T(a)|\psi\rangle = \psi(x-a) \quad T(a)^\dagger \hat{x} T(a) = \hat{x} + a \quad (11)$$

- Rotations: $U(\chi, \hat{n}) = \exp(-i\chi\vec{L}\cdot\hat{n}/\hbar)$
- Symmetries: $U^\dagger H U = H$ or $[H, \hat{G}] = 0$
- Time evolution (Schrodinger picture):

$$U(t) = \exp(-it\hat{H}/\hbar) \quad |\psi_S(t)\rangle = U(t)|\psi(0)\rangle \quad (12)$$

- Heisenberg picture:

$$\mathcal{O}_H(t) = U(t)^\dagger \mathcal{O}(0) U(t) \quad (13)$$

3. Perturbation Theory (Time independent)

$$H = H_0 + \lambda V \quad (14)$$

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle \quad (15)$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \quad (16)$$

- Non-degenerate case:

$$E_n^{(1)} = \langle \psi_n^{(0)} | V | \psi_n^{(0)} \rangle \quad (17)$$

$$|\psi_n^{(1)}\rangle = - \sum_{k \neq n} |\psi_k^{(0)}\rangle \frac{\langle \psi_k^{(0)} | V | \psi_n^{(0)} \rangle}{E_k^{(0)} - E_n^{(0)}} \quad (18)$$

$$E_n^{(2)} = \langle \psi_n^{(0)} | V | \psi_n^{(1)} \rangle \quad (19)$$

- Degenerate case: for fixed n work in the degenerate subspace spanned by $|\psi_{n,i}^{(0)}\rangle$ for $i = 1, \dots, N_n$ where N_n is the degenerate of the $E = E_n^{(0)}$ subspace. Find the matrix:

$$\mathbb{W}_{ij} = \langle \psi_{n,i}^{(0)} | V | \psi_{n,j}^{(0)} \rangle \quad (20)$$

Then the eigenvalues δE_k give the corrections $E_{n,k}^{(1)} = \delta E_k$ and the “good” eigenstates are $|\tilde{\psi}_{n,k}^{(0)}\rangle = \sum_i (\alpha_k)_i |\psi_{n,i}^{(0)}\rangle$ where $\vec{\alpha}_k$ are the normalized eigenvectors of \mathbb{W} with eigenvalue δE_k .