

Here are some optional problems on integral equations. They are taken *verbatim* from Paul Goldbart's homework sets.

1) Integral equations:

- a) Solve the inhomogeneous type II Fredholm integral equation

$$u(x) = e^x + \lambda \int_0^1 xy u(y) dy.$$

- b) Solve the homogeneous type II Fredholm integral equation

$$u(x) = \lambda \int_0^\pi \sin(x-y) u(y) dy.$$

- c) Solve the inhomogeneous type II Fredholm integral equation

$$u(x) = x + \lambda \int_0^1 y(x+y) u(y) dy$$

to second order in λ using

- i) the Liouville-Neumann-Born series; and
 - ii) the Fredholm series.
- d) By differentiating, solve the integral equation: $u(x) = x + \int_0^x u(y) dy$.
- e) Solve the integral equation: $u(x) = x^2 + \int_0^1 xy u(y) dy$.
- f) Find the eigenfunction(s) and eigenvalue(s) of the integral equation

$$u(x) = \lambda \int_0^1 e^{x-y} u(y) dy.$$

- g) Solve the integral equation: $u(x) = e^x + \lambda \int_0^1 e^{x-y} u(y) dy$.

2) Neumann Series: Consider the integral equation

$$u(x) = g(x) + \lambda \int_0^1 K(x, y) u(y) dy,$$

in which only u is considered unknown.

- a) Write down the solution $u(x)$ to second order in the Liouville-Neumann-Born series.
- b) Suppose $g(x) = x$ and $K(x, y) = \sin 2\pi xy$. Compute $u(x)$ to second order in the Liouville-Neumann-Born series. (You may leave your answer to the second-order term in terms of a single integral.)

3) Translationally invariant kernels:

- a) Consider the integral equation: $u(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y) u(y) dy$, with the translationally invariant kernel $K(x, y) = Q(x-y)$, in which g , λ and Q are considered known. Show that the Fourier transforms \hat{u} , \hat{g} and \hat{Q} satisfy $\hat{u}(q) = \hat{g}(q)/\{1 - \sqrt{2\pi}\lambda\hat{Q}(q)\}$. Expand this result to second order in λ to recover the second-order Liouville-Neumann-Born series.
- b) Use Fourier transforms to find a solution of the integral equation

$$u(x) = e^{-|x|} + \lambda \int_{-\infty}^{\infty} e^{-|x-y|} u(y) dy$$

which remains finite as $|x| \rightarrow \infty$.

- c) Use Laplace transforms to find a solution for $x > 0$ of the integral equation

$$u(x) = e^{-x} + \lambda \int_0^x e^{-|x-y|} u(y) dy.$$