Physics 508	**Mathematical Methods in Physics I**	Prof. M. Stone
Handout 2	Course Material	2117 ESB
Fall 2016	HOMEWORK 2	University of Illinois

1) Fermat's principle: According to Fermat's principle, the path taken by a ray of light between any two points makes stationary the travel time between those points. A medium is characterised optically by its *refractive index* n, such that the speed of light in the medium is c/n. In this problem we will assume that all light rays lie in the $x-y$ plane.

a) Use Fermat's principle to show that light propagates along straight lines in homogeneous media (*i.e.*, media in which n is independent of position).

b) Consider the propagation of light from one semi-infinite homogeneous medium of refractive index n_1 to another with refractive index n_2. By examining paths that need not be differentiable at the interface, establish Snell's law.

c) A planar light ray propagates in an layered medium with refractive index $n(x)$. Use Fermat's principle to establish a generalized Snell's law in the form $n \sin\psi = \text{constant}$ by finding the equation for stationary paths for

$$F_1 = \int n(x)\sqrt{1+y'^2}dx.$$

(Here the prime denotes differentiation with respect to x.) Repeat this exercise with $x \leftrightarrow y$ by finding a similar equation for the stationary paths of

$$F_2 = \int n(y)\sqrt{1+y'^2}dx.$$

By using suitable definitions of the angle of incidence ψ, in each case show that the two formulations of the problem of a layered medium give physically equivalent answers. (In the second formulation you will find it easiest to use the first integral of Euler's equation.)

2) Hyperbolic Geometry: This problem involves a version of the Poincaré model for the non-Euclidean geometry of Lobachevski.

a) Show that the stationary paths for the functional

$$F_3 = \int \frac{1}{y}\sqrt{1+y'^2}dx,$$

with y restricted to lying in the upper half plane are circles of arbitrary radius and with centres on the x axis. These paths are the *geodesics*, or minimum length paths, in a space with Riemann metric

$$ds^2 = \frac{1}{y^2}(dx^2+dy^2), \qquad y>0$$

b) Show that if we call these geodesics "lines", then one and only one line can be drawn though two given points.

c) Two lines are said to be *parallel* if, and only if, they meet at "infinity", *i.e.* on the x axis. (Verify that the x axis is indeed infinitely far from any point with $y > 0$.) Show that given a line q and a point A not lying on that line, that there are *two* lines passing through A that are parallel to q, and that between these two lines lies a pencil of lines passing through A that never meet q.

3) Drums and Membranes: The shape of a distorted drumskin is described by the function $h(x, y)$, which gives the height to which the point (x, y) of the flat undistorted drumskin is displaced.

a) Show that the area of the distorted drumskin is equal to

$$\text{Area} = \int dx\, dy \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2},$$

where the integral is taken over the area of the flat drumskin.

b) Show that for small distortions, the area reduces to

$$\mathcal{A}[h] = \text{const.} + \frac{1}{2}\int dx\, dy\, |\nabla h|^2,$$

where $\nabla \equiv \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y$.

c) Show that if h satisfies the two-dimensional Laplace equation then $\mathcal{A}$ is stationary with respect to variations that vanish at the boundary.

d) Suppose the drumskin has mass ρ_0 per unit area, and surface tension T. Write down the Lagrangian controlling the motion of the drumskin, and derive the equation of motion that follows from it.

4) Magnetostatics: We wish to find the magnetic field $\mathbf{B}(x)$, $x \in \mathbb{R}^3$, produced by a (compactly supported) current distribution $\mathbf{J}(x)$ in a material with position-dependent permeability $\mu(x)$. Consider the functional

$$F[\mathbf{A}] = \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu(x)} |\nabla \times \mathbf{A}|^2 - \mathbf{J} \cdot \mathbf{A} \right\} d^3x,$$

where $\mathbf{A}(x)$ is a candidate vector potential for the field $\mathbf{B} \equiv \nabla \times \mathbf{A}$.

i) Show that $F[\mathbf{A}]$ is *gauge invariant* (*i.e.* it is unchanged in value when $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\chi$) provided that $\nabla \cdot \mathbf{J} = 0$.

ii) Show that the stationarity condition $\delta F/\delta A_i(x) = 0$ leads to $\mathbf{A}$ obeying the appropriate Maxwell equation. (You may assume that $\nabla \times \mathbf{A}$ is zero at infinity.)

iii) By "completing the square," show that $F[\mathbf{A}]$ takes its *minimum* value when $\mathbf{A}$ is a vector potential for the actual magnetic field produced by the current distribution $\mathbf{J}$.