Physics 508 | **Mathematical Methods in Physics I** | Prof. M. Stone

Handout 4 | Course Material | 2117 ESB

Fall 2016 | Homework 4 | University of Illinois

1) Test functions and distributions:

a) Let $f(x)$ be a smooth function.

i) Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that

$$\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x).$$

ii) We might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x).$$

By integrating both against a test function, show this expression for the derivative of $f(x)\delta(x)$ is equivalent to that in part i).

b) Let $G(x)$ be a smooth function that decreases rapidly to zero as $|x| \to \infty$, and $\varphi(x)$ a smooth function such that its derivative $\varphi'(x)$ decreases rapidly to zero as $|x| \to \infty$. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi'(x)\varphi'(y)G(|x-y|)\,dxdy = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\varphi(x)-\varphi(y)]^2 G''(|x-y|)\,dxdy.$$

c) In a paper[1] that has recently been cited in the literature on topological insulators a distribution $\delta^{(1/2)}(x)$ is defined by setting

$$\delta^{(1/2)}(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}|k|^{1/2}e^{ikx}.$$

The Fourier transform on the RHS is clearly divergent, so we need to decide how to interpret it. Let's try to define the evaluation of $\delta^{(1/2)}$ on a test function $\varphi(x)$ as

$$\int_{-\infty}^{\infty}\delta^{(1/2)}(x)\varphi(x)\,dx \stackrel{\text{def}}{=} \lim_{\mu\to 0_+}\left\{\int_{-\infty}^{\infty}\delta_\mu^{(1/2)}(x)\varphi(x)\,dx\right\}.$$

where

$$\begin{aligned}\delta_\mu^{(1/2)}(x) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{ikx}|k|^{1/2}e^{-\mu|k|}\,\frac{dk}{2\pi}\\ &= \sqrt{\frac{1}{4\pi}}(x^2+\mu^2)^{-3/4}\cos\left(\frac{3}{2}\tan^{-1}\left(\frac{x}{\mu}\right)\right).\end{aligned}$$

(Could you have evaluated this integral if I had not given you the answer?)

[1] H. Aratyn, *Fermions from Bosons in 2+1 dimensions*, Phys. Rev. D **28** (1983) 2016-18.

Plot some graphs of $\delta_\mu^{(1/2)}(x)$ for various values of μ, and so get an idea of how it behaves as the convergence factor $e^{-\mu|k|} \to 1$. Deduce that

$$\int_{-\infty}^{\infty} \delta^{(1/2)}(x)\varphi(x)\,dx = -\sqrt{\frac{1}{8\pi}}\int_{-\infty}^{\infty} \frac{1}{|x|^{3/2}}\{\varphi(x)-\varphi(0)\}\,dx.$$

(Hint: Observe that $\delta_\mu^{(1/2)}(x)$ is the Fourier transform of a function that vanishes at $k=0$. What property of the the graph of $\delta_\mu^{(1/2)}(x)$ does this imply?)

d) Let $\varphi(x)$ be a test function. Using the definition of the *principal part integral*, show that

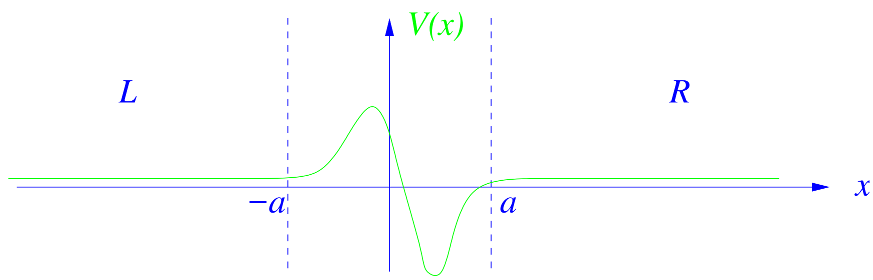
$$\frac{d}{dt}\left\{P\int_{-\infty}^{\infty}\frac{\varphi(x)}{(x-t)}\,dx\right\} = P\int_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,dx$$

To do this fix the value of the cutoff ϵ and then differentiate the resulting ϵ-regulated integral, taking care to include the terms arising from the t dependence of the limits at $x = t \pm \epsilon$.

2) One-dimensional scattering theory: Consider the one-dimensional Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

where $V(x)$ is zero except in a finite interval $[-a,a]$ near the origin.



Let L denote the left asymptotic region, $-\infty < x < -a$, and similarly let R denote $\infty > x > a$. For $E = k^2$ and $k > 0$ there will be scattering solutions of the form

$$\psi_k(x) = \begin{cases} e^{ikx} + r_L(k)e^{-ikx}, & x \in L, \\ t_L(k)e^{ikx}, & x \in R, \end{cases}$$

describing waves incident on the potential $V(x)$ from the left. For $k < 0$ there will be solutions with waves incident from the right

$$\psi_k(x) = \begin{cases} t_R(k)e^{ikx}, & x \in L, \\ e^{ikx} + r_R(k)e^{-ikx}, & x \in R. \end{cases}$$

The wavefunctions in $[-a,a]$ will naturally be more complicated. Observe that $[\psi_k(x)]^*$ is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:

a) $|r_{L,R}|^2 + |t_{L,R}|^2 = 1$,

b) $t_L(k)=t_R(-k)$.

c) Deduce from parts a) and b) that $|r_L(k)| = |r_R(-k)|$.

d) Take the specific example of $V(x) = \lambda\delta(x-b)$ with $|b| < a$. Compute the transmission and reflection coefficients and hence show that $r_L(k)$ and $r_R(-k)$ may differ by a phase.

3) Reduction of Order: Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that $y_1 = u(x)$ is one solution to the equation

$$y'' + V(x)y = 0.$$

By trying $y = u(x)v(x)$ show that

$$y_2 = u(x)\int^x \frac{d\xi}{u^2(\xi)}$$

is also a solution of the differential equation. Is this new solution ever merely a constant mutiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian $W(y_2, y_1)$.)

b) Suppose that we are told that the product, y_1y_2, of the two solutions to the equation $y'' + p_1y' + p_2y = 0$ is a constant. Show that this requires $2p_1p_2 + p_2' = 0$.

c) By using ideas from part b) or otherwise, find the general solution of the equation

$$(x+1)x^2y'' + xy' - (x+1)^3y = 0.$$

4) Normal forms and the Schwarzian derivative: We saw in class that if y obeys a second-order linear differential equation

$$y'' + p_1y' + p_2y = 0$$

then we can make always make a substitution $y = w\tilde{y}$ so that $\tilde{y}$ obeys an equation without a first derivative:

$$\tilde{y}'' + q(x)\tilde{y} = 0.$$

Suppose $\psi(x)$ obeys a Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + [V(x) - E]\right)\psi = 0.$$

a) Make a smooth and invertible change of independent variable by setting $x = x(z)$ and find the second order differential equation in z obeyed by $\psi(z) \equiv \psi(x(z))$. Find the $\tilde{\psi}(z)$ that obeys an equation with no first derivative. Show that this equation is

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} + (x')^2[V(x(z)) - E] - \frac{1}{4}\{x, z\} \right) \tilde{\psi}(z) = 0,$$

where the primes denote differentiation with respect to z, and

$$\{x, z\} \equiv \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2$$

is called the *Schwarzian* derivative of x with respect to z. Schwarzian derivatives play an important role in conformal field theory and string theory.

b) Now combine a sequence of maps $x \to z \to w$ to establish *Cayley's identity*

$$\left(\frac{dz}{dw}\right)^2 \{x, z\} + \{z, w\} = \{x, w\}.$$

(Hint: If this takes you more than a line or two, you are missing the point of the problem.)