

Here are some optional problems on integral equations. They are taken *verbatim* from Paul Goldbart's homework sets.

**1) Integral equations:**

- a) Solve the inhomogeneous type II Fredholm integral equation

$$u(x) = e^x + \lambda \int_0^1 xy u(y) dy.$$

- b) Solve the homogeneous type II Fredholm integral equation

$$u(x) = \lambda \int_0^\pi \sin(x-y) u(y) dy.$$

- c) Solve the inhomogeneous type II Fredholm integral equation

$$u(x) = x + \lambda \int_0^1 y(x+y) u(y) dy$$

to second order in  $\lambda$  using

- i) the Liouville-Neumann-Born series; and
  - ii) the Fredholm series.
- d) By differentiating, solve the integral equation:  $u(x) = x + \int_0^x u(y) dy$ .
- e) Solve the integral equation:  $u(x) = x^2 + \int_0^1 xy u(y) dy$ .
- f) Find the eigenfunction(s) and eigenvalue(s) of the integral equation

$$u(x) = \lambda \int_0^1 e^{x-y} u(y) dy.$$

- g) Solve the integral equation:  $u(x) = e^x + \lambda \int_0^1 e^{x-y} u(y) dy$ .

**2) Neumann Series:** Consider the integral equation

$$u(x) = g(x) + \lambda \int_0^1 K(x, y) u(y) dy,$$

in which only  $u$  is considered unknown.

- a) Write down the solution  $u(x)$  to second order in the Liouville-Neumann-Born series.
- b) Suppose  $g(x) = x$  and  $K(x, y) = \sin 2\pi xy$ . Compute  $u(x)$  to second order in the Liouville-Neumann-Born series. (You may leave your answer to the second-order term in terms of a single integral.)

**3) Translationally invariant kernels:**

- a) Consider the integral equation:  $u(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y) u(y) dy$ , with the translationally invariant kernel  $K(x, y) = Q(x - y)$ , in which  $g$ ,  $\lambda$  and  $Q$  are considered known. Show that the Fourier transforms  $\hat{u}$ ,  $\hat{g}$  and  $\hat{Q}$  satisfy  $\hat{u}(q) = \hat{g}(q) / \{1 - \sqrt{2\pi}\lambda\hat{Q}(q)\}$ . Expand this result to second order in  $\lambda$  to recover the second-order Liouville-Neumann-Born series.
- b) Use Fourier transforms to find a solution of the integral equation

$$u(x) = e^{-|x|} + \lambda \int_{-\infty}^{\infty} e^{-|x-y|} u(y) dy$$

which remains finite as  $|x| \rightarrow \infty$ .

- c) Use Laplace transforms to find a solution for  $x > 0$  of the integral equation

$$u(x) = e^{-x} + \lambda \int_0^x e^{-|x-y|} u(y) dy.$$