1) **Fermat’s principle:** According to Fermat’s principle, the path taken by a ray of light between any two points makes stationary the travel time between those points. A medium is characterised optically by its *refractive index* $n$, such that the speed of light in the medium is $c/n$. In this problem we will assume that all light rays lie in the $x – y$ plane.

a) Use Fermat’s principle to show that light propagates along straight lines in homogeneous media (i.e., media in which $n$ is independent of position).

b) Consider the propagation of light from a flat slab of glass of refractive index $n_1$ into to another with refractive index $n_2$. By examining paths that need not be differentiable at the flat interface, establish Snell’s law: $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

c) A planar light ray propagates in an layered medium with refractive index $n(x)$. Use Fermat’s principle to establish a generalized Snell’s law in the form $n \sin \psi = \text{constant}$ by finding the equation for stationary paths for

$$F_1 = \int n(x) \sqrt{1 + y'^2} dx.$$  

(Here the prime denotes differentiation with respect to $x$.) Repeat this exercise with $x \leftrightarrow y$ by finding a similar equation for the stationary paths of

$$F_2 = \int n(y) \sqrt{1 + y'^2} dx.$$  

By using suitable definitions of the angle of incidence $\psi$, in each case show that the two formulations of the problem of a layered medium give physically equivalent answers. (In the second formulation you will find it easiest to use the first integral of Euler’s equation.)

2) **Hyperbolic Geometry:** This problem involves a version of the Poincaré model for the non-Euclidean geometry of Lobachevski.

a) Show that the stationary paths of the functional

$$F_3 = \int \frac{1}{y} \sqrt{1 + y'^2} dx,$$

with $y$ restricted to lying in the upper half plane are circles of arbitrary radius and with centres on the $x$ axis. These paths are the *geodesics*, or minimum length paths, in a space with Riemann metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2), \quad y > 0$$
b) Show that if we call these geodesics “lines”, then one and only one line can be drawn through two given points.

c) Two lines are said to be parallel if, and only if, they meet at “infinity”, i.e. on the $x$ axis. (Verify that the $x$ axis is indeed infinitely far from any point with $y > 0$.) Show that given a line $q$ and a point $A$ not lying on that line, that there are two lines passing through $A$ that are parallel to $q$, and that between these two lines lies a pencil of lines passing through $A$ that never meet $q$.

3) Drums and Membranes: The shape of a distorted drumskin is described by the function $h(x, y)$, which gives the height to which the point $(x, y)$ of the flat undistorted drumskin is displaced.

   a) Show that the area of the distorted drumskin is equal to

$$\text{Area} = \int dx \, dy \sqrt{1 + \left( \frac{\partial h}{\partial x} \right)^2 + \left( \frac{\partial h}{\partial y} \right)^2},$$

where the integral is taken over the area of the flat drumskin.

   b) Show that for small distortions, the area reduces to

$$\mathcal{A}[h] = \text{const.} + \frac{1}{2} \int dx \, dy \, |\nabla h|^2,$$

where $\nabla \equiv e_x \partial_x + e_y \partial_y$.

   c) Show that if $h$ satisfies the two-dimensional Laplace equation then $\mathcal{A}$ is stationary with respect to variations that vanish at the boundary.

   d) Suppose the drumskin has mass $\rho_0$ per unit area, and surface tension $T$. Write down the Lagrangian controlling the motion of the drumskin, and derive the equation of motion that follows from it.

4) Magnetostatics: We wish to find the magnetic field $B(x)$, $x \in \mathbb{R}^3$, produced by a (compactly supported) current distribution $J(x)$ in a material with position-dependent permeability $\mu(x)$. Consider the functional

$$F[A] = \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu(x)} |\nabla \times A|^2 - J \cdot A \right\} d^3x,$$

where $A(x)$ is a candidate vector potential for the field $B \equiv \nabla \times A$.

   i) Show that $F[A]$ is gauge invariant (i.e. it is unchanged in value when $A$ is replaced by $A + \nabla \chi$) provided that $\nabla \cdot J = 0$.

   ii) Show that the stationarity condition $\delta F/\delta A_i(x) = 0$ leads to $A$ obeying the appropriate Maxwell equation. (You may assume that $\nabla \times A$ is zero at infinity.)

   iii) By “completing the square,” show that $F[A]$ takes its minimum value when $A$ is a vector potential for the actual magnetic field produced by the current distribution $J$. 
