1) Test functions and distributions:

a) Let $f(x)$ be a smooth function.

i) Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that

$$\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x).$$

ii) We might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x).$$

By integrating both against a test function, show this expression for the derivative of $f(x)\delta(x)$ is equivalent to that in part i).

b) In a paper\textsuperscript{1} that has recently been cited in the literature on topological insulators a distribution $\delta^{(1/2)}(x)$ is defined by setting

$$\delta^{(1/2)}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^{1/2} e^{ikx}. $$

The Fourier transform on the RHS is clearly divergent, so we need to decide how to interpret it. Let’s try to define the evaluation of $\delta^{(1/2)}$ on a test function $\varphi(x)$ as

$$\int_{-\infty}^{\infty} \delta^{(1/2)}(x)\varphi(x) = \lim_{\mu \to 0^+} \left\{ \int_{-\infty}^{\infty} \delta^{(1/2)}_{\mu}(x)\varphi(x) dx \right\}. $$

where

$$\delta^{(1/2)}_{\mu}(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{ikx} k^{1/2} e^{-\mu |k|} \frac{dk}{2\pi} = \sqrt{\frac{1}{4\pi}} (x^2 + \mu^2)^{-3/4} \cos \left( \frac{3}{2} \tan^{-1} \left( \frac{x}{\mu} \right) \right). $$

(Could you have evaluated this integral if I had not given you the answer?)

Plot some graphs of $\delta^{(1/2)}_{\mu}(x)$ for various values of $\mu$, and so get an idea of how it behaves as the convergence factor $e^{-\mu |k|} \to 1$. Deduce that

$$\int_{-\infty}^{\infty} \delta^{(1/2)}_{\mu}(x)\varphi(x) dx = -\sqrt{\frac{1}{8\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{3/2}} \{ \varphi(x) - \varphi(0) \} dx.$$ 

(Hint: Observe that $\delta^{(1/2)}_{\mu}(x)$ is the Fourier transform of a function that vanishes at $k = 0$. What property of the the graph of $\delta^{(1/2)}_{\mu}(x)$ does this imply?)

c) Let \( \varphi(x) \) be a test function. Using the definition of the principal part integral, show that
\[
\frac{d}{dt} \left\{ P \int_{-\infty}^{\infty} \frac{\varphi(x)}{x-t} \, dx \right\} = P \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(t)}{(x-t)^2} \, dx
\]
To do this fix the value of the cutoff \( \epsilon \) and then differentiate the resulting \( \epsilon \)-regulated integral, taking care to include the terms arising from the \( t \) dependence of the limits at \( x = t \pm \epsilon \).

2) One-dimensional scattering theory: Consider the one-dimensional Schrödinger equation
\[
-\frac{d^2 \psi}{dx^2} + V(x)\psi = E\psi
\]
where \( V(x) \) is zero except in a finite interval \([ -a, a ]\) near the origin.

\[
\begin{align*}
L & \quad \quad \quad \quad V(x) \quad \quad \quad \quad R \\
-\infty & < x < -a, \quad \quad \quad \quad a < x < \infty
\end{align*}
\]

Let \( L \) denote the left asymptotic region, \(-\infty < x < -a\), and similarly let \( R \) denote \( \infty > x > a \). For \( E = k^2 \) and \( k > 0 \) there will be scattering solutions of the form
\[
\psi_k(x) = \begin{cases} 
    e^{ikx} + r_L(k)e^{-ikx}, & x \in L, \\
    t_L(k)e^{ikx}, & x \in R,
\end{cases}
\]
describing waves incident on the potential \( V(x) \) from the left. For \( k < 0 \) there will be solutions with waves incident from the right
\[
\psi_k(x) = \begin{cases} 
    t_R(k)e^{ikx}, & x \in L, \\
    e^{ikx} + r_R(k)e^{-ikx}, & x \in R.
\end{cases}
\]
The wavefunctions in \([-a, a]\) will naturally be more complicated. Observe that \( [\psi_k(x)]^* \) is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:

a) \( |r_{L,R}|^2 + |t_{L,R}|^2 = 1 \),
b) \( t_L(k) = t_R(-k) \).
c) Deduce from parts a) and b) that \( |r_L(k)| = |r_R(-k)| \).
d) Take the specific example of \( V(x) = \lambda \delta(x-b) \) with \( |b| < a \). Compute the transmission and reflection coefficients and hence show that \( r_L(k) \) and \( r_R(-k) \) may differ by a phase.
3) **Reduction of Order:** Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that \( y_1 = u(x) \) is one solution to the equation

\[
y'' + V(x)y = 0.
\]

By trying \( y = u(x)v(x) \) show that

\[
y_2 = u(x) \int^x \frac{d\xi}{u^2(\xi)}
\]

is also a solution of the differential equation. Is this new solution ever merely a constant multiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian \( W(y_2, y_1) \).)

b) Suppose that we are told that the product, \( y_1y_2 \), of the two solutions to the equation \( y'' + p_1y' + p_2y = 0 \) is a constant. Show that this requires \( 2p_1p_2 + p_1'^2 = 0 \).

c) By using ideas from part b) or otherwise, find the general solution of the equation

\[
(x + 1)x^2y'' + xy' - (x + 1)^3y = 0.
\]

4) **Normal forms and the Schwarzian derivative:** We saw in class that if \( y \) obeys a second-order linear differential equation

\[
y'' + p_1y' + p_2y = 0
\]

then we can make always make a substitution \( y = w\tilde{y} \) so that \( \tilde{y} \) obeys an equation without a first derivative:

\[
\tilde{y}'' + q(x)\tilde{y} = 0.
\]

Suppose \( \psi(x) \) obeys a Schrödinger equation

\[
\left(-\frac{1}{2} \frac{d^2}{dx^2} + [V(x) - E]\right)\psi = 0.
\]

a) Make a smooth and invertible change of independent variable by setting \( x = x(z) \) and find the second order differential equation in \( z \) obeyed by \( \psi(z) \equiv \psi(x(z)) \). Find the \( \tilde{\psi}(z) \) that obeys an equation with no first derivative. Show that this equation is

\[
\left(-\frac{1}{2} \frac{d^2}{dz^2} + (x')^2[V(x(z)) - E] - \frac{1}{4}\{x, z\}\right)\tilde{\psi}(z) = 0,
\]

where the primes denote differentiation with respect to \( z \), and

\[
\{x, z\} \equiv \frac{x''}{x'} - \frac{3}{2} \left(\frac{x''}{x'}\right)^2
\]

is called the **Schwarzian** derivative of \( x \) with respect to \( z \). Schwarzian derivatives play an important role in conformal field theory and string theory.
b) Now combine a sequence of maps $x \rightarrow z \rightarrow w$ to establish *Cayley’s identity*

$$
\left( \frac{dz}{dw} \right)^2 \{x, z\} + \{z, w\} = \{x, w\}.
$$

(Hint: If this takes you more than a line or two, or you find yourself using the hideous expression for \{x, z\}, you are missing the point of the problem.)