Physics 508 | Prof. P. Draper
Homework 1 | **Mathematical Methods in Physics I** | 435 Loomis
Fall 2021 | Homework 1 | University of Illinois

These problems are designed to reactivate your mathematical skillset after a relaxing summer break. They are not designed to be too easy!

Differential Calculus: After taking a previous version of this course, a student claimed that the expression

$$y(x) = \frac{1}{\omega \sin \omega L} \left\{ \int_0^x f(t) \sin \omega(x-L) \sin \omega t \, dt + \int_x^L f(t) \sin \omega x \sin \omega(t-L) \, dt \right\}$$

is the solution to the problem: "Find $y(x)$ obeying the differential equation

$$\frac{d^2 y}{dx^2} + \omega^2 y = f(x)$$

on the interval $[0, L]$ and satisfying the boundary conditions $y(0) = 0 = y(L)$." First examine her solution to see if it obeys the boundary conditions. Then, by differentiating her solution twice with respect to x and substituting the result into the differential equation, investigate whether her solution is correct.

A second student claimed that if $K(x)$ and $f(x)$ are smooth functions with $K(0) = 1$, and we set

$$F(x) = \int_0^x K(x-y) f(y) \, dy,$$

then

$$F'(x) = \int_0^x K(x-y) f'(y) \, dy.$$

(Here the "prime" denotes differentiation with respect to x or y, as appropriate.) Was he right? If so, explain why. If not, find an extra condition on $f(x)$ that makes his equation correct.

Integral Calculus: Let $\mu > \lambda > 0$ be real numbers. Sketch by hand a graph of the function

$$F(t) = \frac{e^{-\lambda t} - e^{-\mu t}}{t}, \quad 0 < t < \infty.$$

Now a student wishes to evaluate the integral

$$I(\lambda, \mu) = \int_0^\infty \frac{e^{-\lambda t} - e^{-\mu t}}{t} \, dt.$$

He breaks it up as

$$I(\lambda, \mu) = \int_0^\infty \frac{e^{-\lambda t}}{t} \, dt - \int_0^\infty \frac{e^{-\mu t}}{t} \, dt.$$

In the first integral he makes the substitution $x = \lambda t$. In the second he sets $x = \mu t$. He ends up with

$$I(\lambda, \mu) = \int_0^\infty \frac{e^{-x}}{x}\, dx - \int_0^\infty \frac{e^{-x}}{x}\, dx.$$

As the two integrals are identical, he concludes that they must cancel and so $I(\lambda, \mu) = 0$. From your sketch of the function being integrated, you know that he has gone wrong somewhere. Locate the error in his method, and make the small but crucial modification that leads to the correct answer. Confirm your result by using Feynman's trick of first computing the easy integral for $\partial I/\partial \mu$, and then integrating the result with respect to μ.

Now use the same "small modification" technique at *both* ends of the integration interval, and so easily evaluate the intimidating-looking integral

$$I = \int_0^\infty \ln\left\{\frac{a + be^{-px}}{a + be^{-qx}}\right\} \frac{dx}{x}.$$

Assume that a, b, p, q are positive real numbers.

Partial Derivatives: Suppose that you know a wavefunction $\psi(x, t)$ obeying the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi.$$

Now start running at velocity $-U$. From your viewpoint the potential is moving past you, so in the moving frame the wavefunction $\tilde{\psi}(x, t)$ must obey the Schrödinger equation

$$i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \tilde{\psi}}{\partial x^2} + V(x - Ut)\tilde{\psi}.$$

How is $\tilde{\psi}$ related to ψ? To find out, make a change of variables

$$\begin{aligned} z &= x - Ut, \\ \tau &= t, \end{aligned}$$

and use the chain rule for partial derivatives to express

$$\left(\frac{\partial \tilde{\psi}}{\partial t}\right)_x, \left(\frac{\partial \tilde{\psi}}{\partial x}\right)_t, \quad \text{in terms of} \quad \left(\frac{\partial \tilde{\psi}}{\partial \tau}\right)_z, \left(\frac{\partial \tilde{\psi}}{\partial z}\right)_\tau.$$

Use these relations to find the equation obeyed by $\tilde{\psi}$ as a function of z and τ. Show that you can obtain a solution $\tilde{\psi}(z, \tau)$ to this new equation by multiplying the original ψ by a phase factor $e^{i\phi(z,\tau)}$. Restore x and t to show that the solution to the equation with the moving potential is

$$\tilde{\psi}(x, t) = e^{imUx/\hbar - i\frac{1}{2}mU^2 t/\hbar}\psi(x - Ut, t).$$

This $\tilde{\psi}(x,t)$ must be how the original wavefunction appears when seen from the moving frame. Evidently, Schrödinger wavefunctions do not transform as scalars under Galilean transformations (i.e., $\tilde{\psi}(x,t) \neq \psi(x - Ut, t)$).

Matrix Algebra: Let V be an N-dimensional vector space, where $N > 1$. Consider a linear operator $T : V \to V$ which, in some basis, is represented by an $N \times N$ matrix. T obeys the equation

$$(T - \lambda I)^p = 0,$$

with $p = N$, but does not obey this equation for any integer $p < N$. Here λ is a real number and I is the identity operator.

i) Show that if T possesses an eigenvector, the corresponding eigenvalue must be λ. Deduce that T *cannot* be diagonalized.

ii) Show that there exists a vector $\mathbf{e}_1$ such that $(T - \lambda I)^N \mathbf{e}_1 = 0$, but no lesser power of $(T - \lambda I)$ kills $\mathbf{e}_1$.

iii) Define $\mathbf{e}_2 = (T - \lambda I)\mathbf{e}_1$, $\mathbf{e}_3 = (T - \lambda I)^2 \mathbf{e}_1$, *etc.* up to $\mathbf{e}_N$. Show that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ are linearly independent. Hint: linear independence means there is no nontrivial solution for $\{c_i\}$ to $\sum_{i=1}^N c_i \mathbf{e}_i = 0$.

iv) Use $\mathbf{e}_1, \ldots, \mathbf{e}_N$ as a basis for your vector space. Taking

$$\mathbf{e}_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_N = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

write out the matrix representing T in the $\mathbf{e}_i$ basis.