Physics 508 | Prof. P. Draper
Homework 2 | **Mathematical Methods in Physics I** | 435 Loomis
Fall 2021 | HOMEWORK 2 | University of Illinois

1) Fermat's principle: According to Fermat's principle, the path taken by a ray of light between any two points makes stationary the travel time between those points. A medium is characterized optically by its *refractive index* n, such that the speed of light in the medium is c/n. In this problem we will assume that all light rays lie in the $x-y$ plane.

a) Use Fermat's principle to show that light propagates along straight lines in homogeneous media (*i.e.*, media in which n is independent of position).

b) Consider the propagation of light from a flat slab of glass of refractive index n_1 into to another with refractive index n_2. By examining paths that need not be differentiable at the flat interface, establish Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

c) A planar light ray propagates in an layered medium with refractive index $n(x)$. Use Fermat's principle to establish a generalized Snell's law in the form $n \sin\psi =$ constant by finding the equation for stationary paths for

$$F_1 = \int n(x)\sqrt{1+y'^2}dx.$$

(Here the prime denotes differentiation with respect to x.) Repeat this exercise with $x \leftrightarrow y$ by finding a similar equation for the stationary paths of

$$F_2 = \int n(y)\sqrt{1+y'^2}dx.$$

By using suitable definitions of the angle of incidence ψ, in each case show that the two formulations of the problem of a layered medium give physically equivalent answers. (In the second formulation you will find it easiest to use the first integral.)

2) Hyperbolic Geometry: This problem involves a version of the Poincaré model for the non-Euclidean geometry of Lobachevski. You may want to review the history of Euclid's fifth postulate on Wikipedia.

a) Show that the stationary paths for the functional

$$F_3 = \int \frac{1}{y}\sqrt{1+y'^2}dx,$$

with y restricted to lying in the upper half plane are circles of arbitrary radius **with centers on the** x **axis.** These paths are the *geodesics*, or minimum length paths, in a space with Riemann metric

$$ds^2 = \frac{1}{y^2}(dx^2+dy^2), \qquad y>0$$

b) OPTIONAL Show (by giving a compass-and-ruler geometric construction) that if we call these geodesics "lines", then one and only one line can be drawn though two given points.

c) OPTIONAL Two straight lines are said to be *parallel* if, and only if, they meet at "infinity", *i.e.* on the x axis. (Verify that the x axis is indeed infinitely far from any point with $y > 0$.) Given a line q and a point A not lying on that line, show that we can use your method from part (b) to construct *two* lines passing through A that are parallel to q, and that between these two lines lies a pencil of lines passing through A that never meet q. (Google "pencil of lines" if you have not heard the phrase before!)

3) Drums and Membranes: The shape of a distorted drumskin is described by the function $h(x, y)$, which gives the height to which the point (x, y) of the flat undistorted drumskin is displaced.

a) Show that the area of the distorted drumskin is equal to

$$\text{Area} = \int dx\, dy \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2},$$

where the integral is taken over the area of the flat drumskin.

b) Show that for small distortions, the area reduces to

$$\mathcal{A}[h] = \text{const.} + \frac{1}{2} \int dx\, dy\, |\nabla h|^2,$$

where $\nabla \equiv \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y$.

c) Show that if h satisfies the two-dimensional Laplace equation then $\mathcal{A}$ is stationary with respect to variations that vanish at the boundary.

d) Suppose the drumskin has mass ρ_0 per unit area, and surface tension T. Write down the Lagrangian controlling the motion of the drumskin and derive the equation of motion that follows from it.

4) Magnetostatics: We wish to find the magnetic field $\mathbf{B}(x)$, $x \in \mathbb{R}^3$, produced by a (compactly supported) current distribution $\mathbf{J}(x)$ in a material with position-dependent permeability $\mu(x)$. Consider the functional

$$F[\mathbf{A}] = \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu(x)} |\nabla \times \mathbf{A}|^2 - \mathbf{J} \cdot \mathbf{A} \right\} d^3x,$$

where $\mathbf{A}(x)$ is a candidate vector potential for the field $\mathbf{B} \equiv \nabla \times \mathbf{A}$.

i) Show that $F[\mathbf{A}]$ is *gauge invariant* (*i.e.* it is unchanged in value when $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\chi$) provided that $\nabla \cdot \mathbf{J} = 0$.

ii) Show that the stationarity condition $\delta F / \delta A_i(x) = 0$ leads to $\mathbf{A}$ obeying the appropriate Maxwell equation. (You may assume that $\nabla \times \mathbf{A}$ is zero at infinity.)

iii) Replace the current distribution by the physical magnetic field it produces, $\nabla \times \mathbf{H}_{physical} = \mathbf{J}$. By "completing the square," show that $F[\mathbf{A}]$ takes its *minimum* value when $\mathbf{A}$ is a vector potential for the physical magnetic field.