Mathematical Methods in Physics I

HOMEWORK 5

*****Due Wed 10/06 5pm*****

1) Linear differential operators:

a) Let $w(x) > 0$. Consider the differential operator $\hat{L} = id/dx$. Find the formal adjoint of L with respect to the inner product $\langle u|v\rangle_w = \int wu^* v\, dx$, and find the corresponding surface term $Q[u, v]$.

b) Now do the same for the operator $M = d^4/dx^4$, for the case $w = 1$. Find the adjoint boundary conditions defining the domain of $M^\dagger$ for the case

$$\mathcal{D}(M) = \{y, y^{(4)} \in L^2[0, 1] : y(0) = y'''(0) = y(1) = y'''(1) = 0\}.$$

Is M truly self-adjoint? (Hint: you may find the identity

$$f^{(4)}g - fg^{(4)} = \frac{d}{dx}\{f'''g - f''g' + f'g'' - fg'''\}$$

to be of use.)

2) Sturm-Liouville forms: By constructing appropriate weight functions convert the following common operators into Sturm-Liouville form:

a) $\hat{L} = (1 - x^2)\, d^2/dx^2 + [(\mu - \nu) - (\mu + \nu + 2)x]\, d/dx$. (Jacobi)

b) $\hat{L} = (1 - x^2)\, d^2/dx^2 - 3x\, d/dx$. (Chebyshev)

c) $\hat{L} = d^2/dx^2 - 2x(1 - x^2)^{-1}\, d/dx - m^2\, (1 - x^2)^{-1}$. (Legendre)

3) Factorization: Schrödinger equations of the form

$$-\frac{d^2\psi}{dx^2} - l(l + 1)\mathrm{sech}^2 x\, \psi = E\psi$$

are known as *Pöschel-Teller equations*. By setting $u = l \tanh x$ and following the strategy of this problem one may relate solutions for l to those for $l - 1$ and so find all bound states and scattering eigenfunctions for any integer l.

a) Suppose that we know that $\psi = \exp\left\{-\int^x u(x')dx'\right\}$ is a solution of

$$L\psi \equiv \left(-\frac{d^2}{dx^2} + W(x)\right)\psi = 0.$$

Show that L can be written as $L = M^\dagger M$ where

$$M = \left(\frac{d}{dx} + u(x)\right), \quad M^\dagger = \left(-\frac{d}{dx} + u(x)\right),$$

the adjoint being taken with respect to the product $\langle u|v\rangle = \int u^* v\, dx$.

b) Now assume L is acting on functions on $[-\infty, \infty]$ and that we not have to worry about boundary conditions. Show that given an eigenfunction ψ_- obeying $M^\dagger M\psi_- = \lambda\psi_-$ we can multiply this equation on the left by M and so find a eigenfunction ψ_+ with the same eigenvalue for the differential operator

$$L' = MM^\dagger = \left(\frac{d}{dx} + u(x)\right)\left(-\frac{d}{dx} + u(x)\right)$$

and *vice-versa*. Show that this correspondence $\psi_- \leftrightarrow \psi_+$ will fail if, *and only if*, $\lambda = 0$.

c) Apply the strategy from part b) in the case $u(x) = \tanh x$ and one of the two differential operators $M^\dagger M$, $MM^\dagger$ is (up to an additive constant)

$$H = -\frac{d^2}{dx^2} - 2\,\mathrm{sech}^2 x.$$

Show that H has eigenfunctions of the form $\psi_k = e^{ikx}P(\tanh x)$ and eigenvalue $E = k^2$ for any k in the range $-\infty < k < \infty$. The function $P(\tanh x)$ is a polynomial in $\tanh x$ which you should be able to find explicitly. By thinking about the exceptional case $\lambda = 0$, show that H has an eigenfunction $\psi_0(x)$, with eigenvalue $E = -1$, that tends rapidly to zero as $x \to \pm\infty$. Observe that there is no corresponding eigenfunction for the other operator of the pair.