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### Mathematical Methods in Physics I

#### Homework 5

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*****Due Wed 10/06 5pm*****

1) **Linear differential operators:**

   a) Let \( w(x) > 0 \). Consider the differential operator \( \hat{L} = id/dx \). Find the formal adjoint of \( L \) with respect to the inner product \( \langle u|v \rangle_w = \int w u^* v \, dx \), and find the corresponding surface term \( Q[u,v] \).

   b) Now do the same for the operator \( M = d^4/dx^4 \), for the case \( w = 1 \). Find the adjoint boundary conditions defining the domain of \( M^\dagger \) for the case \( \mathcal{D}(M) = \{ y, y^{(4)} \in L^2[0,1] : y(0) = y'''(0) = y(1) = y''''(1) = 0 \} \).

   Is \( M \) truly self-adjoint? (Hint: you may find the identity \( f^{(4)}g - fg^{(4)} = \frac{d}{dx} \{ f'''g - f''g' + f'g'' - fg''' \} \) to be of use.)

2) **Sturm-Liouville forms:** By constructing appropriate weight functions convert the following common operators into Sturm-Liouville form:

   a) \( \hat{L} = (1 - x^2) d^2/dx^2 + [(\mu - \nu) - (\mu + \nu + 2)x] d/dx \). (Jacobi)

   b) \( \hat{L} = (1 - x^2) d^2/dx^2 - 3x d/dx \). (Chebyshev)

   c) \( \hat{L} = d^2/dx^2 - 2x(1 - x^2)^{-1} d/dx - m^2 (1 - x^2)^{-1} \). (Legendre)

3) **Factorization:** Schrödinger equations of the form

   \[ -\frac{d^2\psi}{dx^2} - l(l+1)\text{sech}^2x \psi = E\psi \]

   are known as Pöschel-Teller equations. By setting \( u = l \tanh x \) and following the strategy of this problem one may relate solutions for \( l \) to those for \( l-1 \) and so find all bound states and scattering eigenfunctions for any integer \( l \).

   a) Suppose that we know that \( \psi = \exp \left\{ - \int^x u(x')dx' \right\} \) is a solution of

   \[ L\psi = \left( -\frac{d^2}{dx^2} + W(x) \right) \psi = 0. \]

   Show that \( L \) can be written as \( L = M^\dagger M \) where

   \[ M = \left( \frac{d}{dx} + u(x) \right), \quad M^\dagger = \left( -\frac{d}{dx} + u(x) \right), \]

   the adjoint being taken with respect to the product \( \langle u|v \rangle = \int u^* v \, dx \).
b) Now assume \( L \) is acting on functions on \([-\infty, \infty]\) and that we not have to worry about boundary conditions. Show that given an eigenfunction \( \psi_- \) obeying \( M^\dagger M \psi_- = \lambda \psi_- \) we can multiply this equation on the left by \( M \) and so find a eigenfunction \( \psi_+ \) with the same eigenvalue for the differential operator

\[
L' = MM^\dagger = \left( \frac{d}{dx} + u(x) \right) \left( -\frac{d}{dx} + u(x) \right)
\]

and *vice-versa*. Show that this correspondence \( \psi_- \leftrightarrow \psi_+ \) will fail if, and only if, \( \lambda = 0 \).

c) Apply the strategy from part b) in the case \( u(x) = \tanh x \) and one of the two differential operators \( M^\dagger M, MM^\dagger \) is (up to an additive constant)

\[
H = -\frac{d^2}{dx^2} - 2 \operatorname{sech}^2 x.
\]

Show that \( H \) has eigenfunctions of the form \( \psi_k = e^{ikx} P(\tanh x) \) and eigenvalue \( E = k^2 \) for any \( k \) in the range \(-\infty < k < \infty\). The function \( P(\tanh x) \) is a polynomial in \( \tanh x \) which you should be able to find explicitly. By thinking about the exceptional case \( \lambda = 0 \), show that \( H \) has an eigenfunction \( \psi_0(x) \), with eigenvalue \( E = -1 \), that tends rapidly to zero as \( x \to \pm \infty \). Observe that there is no corresponding eigenfunction for the other operator of the pair.