Solutions to Homework Set 1

**Differential calculus:** The point of the exercise was to make sure that you know how to differentiate integrals with respect to their limits:

\[
\frac{d}{da} \int_a^b f(x) \, dx = -f(a), \quad \frac{d}{db} \int_a^b f(x) \, dx = f(b),
\]

and in general

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} f(x) \, dx = f(b(t)) \frac{\partial b}{\partial t} - f(a(t)) \frac{\partial a}{\partial t} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x, t) \, dx.
\]

Once you have this under control, the first problem is plug and chug:

From

\[
y(x) = \frac{\sin \omega(x - L)}{\omega \sin \omega L} \int_0^x f(t) \sin \omega t \, dt + \frac{\sin \omega x}{\omega \sin \omega L} \int_x^L f(t) \sin \omega(t - L) \, dt
\]

we get

\[
y'(x) = \frac{\cos \omega(x - L)}{\sin \omega L} \int_0^x f(t) \sin \omega t \, dt + \frac{\cos \omega x}{\sin \omega L} \int_x^L f(t) \sin \omega(t - L) \, dt.
\]

The two terms arising from the derivative of the integration limits have cancelled against each other. When we differentiate again, the two pieces arising from differentiating the factors outside the integral assemble to give \(-\omega^2 y(x)\). The two terms from differentiating the integrals are

\[
y''(x) + \omega^2 y(x) = f(x) \frac{\cos \omega(x - L) \sin \omega x - \cos \omega x \sin \omega(x - L)}{\sin \omega L}.
\]

The addition formula for \(\sin(A + B)\), now shows that this is equal to \(f(x)\) as required.

For the second problem we have

\[
F'(x) = K(0)f(x) + \int_0^x \partial_x K(x - y)f(y) \, dy = K(0)f(x) - \int_0^x f(y) \partial_y K(x - y) \, dy
\]

\[
= K(0)f(x) - \int_0^x \partial_y[f(y)K(x - y)] \, dy + \int_0^x f'(y)K(x - y) \, dy
\]

\[
= K(0)f(x) - K(0)f(x) + f(0)K(x) + \int_0^x f'(y)K(x - y) \, dy
\]

\[
= f(0)K(x) + \int_0^x K(x - y)f'(y) \, dy.
\]

He was not quite right therefore—unless \(f(0)\) happens to be zero.
Integral Calculus: His mistake is at the first step:

\[ I(\lambda, \mu) = \int_0^\infty t^{-1} (e^{-\lambda t} - e^{-\mu t}) \, dt = \int_0^\infty t^{-1} e^{-\lambda t} \, dt - \int_0^\infty t^{-1} e^{-\mu t} \, dt. \]

The integrals on the right-hand side are divergent, so this is nonsense. To fix his method we put limits on the divergent integrals:

\[
\int_0^\infty \frac{e^{-\lambda t} - e^{-\mu t}}{t} \, dt = \lim_{\epsilon \to 0} \left\{ \int_\epsilon^\infty \frac{e^{-\lambda t} - e^{-\mu t}}{t} \, dt \right\} = \lim_{\epsilon \to 0} \left\{ \int_\epsilon^\infty \frac{e^{-\lambda t}}{t} \, dt - \int_\epsilon^\infty \frac{e^{-\mu t}}{t} \, dt \right\} = \lim_{\epsilon \to 0} \left\{ \int_\epsilon^\infty \frac{e^{-t}}{t} \, dt - \int_{\mu \epsilon}^\infty \frac{e^{-t}}{t} \, dt \right\} = \lim_{\epsilon \to 0} \left\{ \int_{\lambda \epsilon}^{\mu \epsilon} \frac{1}{s} \, ds \right\} = \int_{\lambda}^{\mu} \frac{1}{s} \, ds = \ln \mu - \ln \lambda.
\]

We could take the limit in the antepenultimate line because \( e^{-\epsilon s} \to 1 \) uniformly in \([\lambda, \mu]\).

Feynman would differentiate under the integral sign:

\[
\frac{d}{d\mu} I(\lambda, \mu) = \int_0^\infty \frac{d}{d\mu} \left( \frac{e^{-\lambda t} - e^{-\mu t}}{t} \right) \, dt = \int_0^\infty e^{-\mu t} \, dt = 1/\mu.
\]

Integrating up, he would find.

\[ I(\lambda, \mu) = \ln \mu + c. \]

He can determine the constant \( c \) by noting that \( I = 0 \) when \( \lambda = \mu \), whence

\[ I(\lambda, \mu) = \ln \mu - \ln \lambda. \]

When \( f \) does not tend to zero at infinity we can use the same scaling trick there to get the general Frulani integral

\[ F(\lambda, \mu) = \int_0^\infty \frac{f(\lambda t) - f(\mu t)}{t} \, dt = (f(0) - f(\infty)) \ln(\mu/\lambda). \]

We apply this to the case \( f(x) = \ln(a + be^{-px}) \) to get

\[ I = \int_0^\infty \ln \left\{ \frac{a + be^{-px}}{a + be^{-qx}} \right\} \frac{dx}{x} = \ln \left\{ \frac{a}{a + b} \right\} \ln \left\{ \frac{p}{q} \right\} \]
This example is found in Gradshteyn and Ryzhik’s *Table of Integrals, Series, and Products* as number 4.319.3.

**Partial derivatives:** Using the chain rule we have

\[
\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial z}{\partial t} \frac{\partial}{\partial z} = \frac{\partial}{\partial \tau} - U \frac{\partial}{\partial z},
\]

and

\[
\frac{\partial}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial}{\partial z} + \frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial z}.
\]

We therefore find that

\[
i\hbar \left( \frac{\partial}{\partial \tau} - U \frac{\partial}{\partial z} \right) \tilde{\psi} = -\frac{\hbar^2}{2m} \frac{\partial^2 \tilde{\psi}}{\partial z^2} + V(z) \tilde{\psi}.
\]

The potential is now time independent. This modified Schrödinger equation is solved by plugging in \( \tilde{\psi} = e^{i(ax + \beta \tau)} \psi(z, \tau) \) and finding \( \alpha \) and \( \beta \).

On plugging in and differentiating, we find that

\[
-\hbar \beta \psi + i\hbar \frac{\partial \psi}{\partial \tau} + \hbar U \alpha \psi - i\hbar U \frac{\partial \psi}{\partial z} = -\frac{\hbar^2}{2m} \left(-\alpha^2 \psi + 2i \alpha \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(z) \psi.
\]

Most of this equation is that satisfied by \( \psi(z, \tau) \). The bits that are left over contain \( \partial \psi / \partial z \), and \( \psi \). From the vanishing of the coefficient of \( \partial \psi / \partial z \) we have

\[
\alpha = \frac{mU}{\hbar}
\]

and, from the vanishing of the coefficient of \( \psi \),

\[
\beta = U \alpha - \frac{\hbar^2}{2m} \alpha^2 = \frac{mU^2}{2\hbar}.
\]

All other terms sum to zero as they are the equation satisfied by \( \psi \). Thus

\[
\tilde{\psi} = e^{imUz/\hbar + i\frac{1}{2}mU^2 \tau / \hbar} \psi(z, \tau).
\]

Resubstituting, \( \tau = t, z = x - Ut \) now leads to

\[
\tilde{\psi}(x, t) = e^{imUt/\hbar - i\frac{1}{2}mU^2 t / \hbar} \psi(x - Ut, t).
\]

The wave-function does not transform as a scalar function, but is instead an observer dependent object.

To check the solution we evaluate the two sides of

\[
i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \tilde{\psi}}{\partial x^2} + V(x - Ut) \tilde{\psi}.
\]
The LHS is
\[ i\hbar \partial_t \tilde{\psi} = i\hbar \left( -\frac{i}{2\hbar} mU^2 - U\psi' + \dot{\psi} \right) e^{imUx/\hbar - i\frac{1}{2}mU^2 t/\hbar} \]
The RHS is
\[ -\frac{\hbar^2}{2m} \left( -\frac{m^2}{\hbar^2} U^2 + 2i\frac{m}{\hbar} U\psi' + \psi'' + V \right) e^{imUx/\hbar - i\frac{1}{2}mU^2 t/\hbar}. \]
We see that these coincide if
\[ i\hbar \dot{\psi} = -\frac{\hbar^2}{2m} \psi'' + V\psi. \]
In all these last three equations \( \psi \) and \( V \) are functions of \( x - Ut, t \), and \( \psi' \) denotes the derivative of \( \psi \) with respect to its first slot.

**Matrix Algebra:**

i) Suppose that \( T \) has an eigenvector \( x \) with eigenvalue \( \mu \), so \( Tx = \mu x \). Then
\[ 0 = (T - \lambda I)^N x = (\mu - \lambda)^N x. \]
Since \( x \) is non-zero we see that \( (\lambda - \mu)^N = 0 \), but if the \( N \)-th power of a number is zero, the number itself must be zero. Thus \( \lambda \) is the only possible eigenvalue. If the matrix representing \( T \) were diagonalizable, all the numbers on the diagonal would have to be \( \lambda \) and the diagonalized matrix would be \( T = \lambda I \). This matrix, and hence the linear operator \( T \) that it represents, would then obey \( (T - \lambda I)^1 = 0 \) — but, unless \( N = 1 \), this is in contradiction to what we were told about \( T \). Thus \( T \) cannot be diagonalized.

ii) That there exists a vector \( e_1 \) such that \( (T - \lambda I)^N e_1 = 0 \), but no lesser power of \( (T - \lambda I) \) kills \( e_1 \), is simply a restatement of what we are given: all vectors are killed by \( (T - \lambda I)^N \), but if there were no vector that survived \( (T - \lambda I)^{N-1} \), then we would have \( (T - \lambda I)^{N-1} = 0 \). We are told, however, that \( (T - \lambda I)^{N-1} \) is not zero.

iii) A set of vectors \( \{e_i\} \) is **linearly independent** if no non-trivial linear combination of the vectors is the zero vector. We therefore need to show that if
\[ \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_N e_N = 0, \]
then all \( N \) of the \( \lambda_i \) are zero.

To do this, act on this expression by \( (T - \lambda I)^{N-1} \). We find
\[ 0 = \lambda_1 (T - \lambda I)^{N-1} e_1, \]
all other terms being killed. Since we know that \( (T - \lambda I)^{N-1} e_1 \neq 0 \), we must have \( \lambda_1 = 0 \). Now act with \( (T - \lambda I)^{N-2} \), to get
\[ 0 = \lambda_2 (T - \lambda I)^{N-2} e_2 = \lambda_2 (T - \lambda I)^{N-1} e_1, \]
all other terms being killed, or being zero because \( \lambda_1 = 0 \). We now deduce that \( \lambda_2 = 0 \). Proceeding in this manner we deduce that all the \( \lambda_i \) are zero. Thus the \( e_i \) are indeed linearly independent.
iv) In the $e_i$ basis the matrix becomes

$$T \to T = \begin{pmatrix} \lambda & 1 & & & \\ \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \end{pmatrix},$$

i.e. a matrix with $\lambda$’s on the diagonal, and 1’s immediately above, all other entries being zero. This is called the Jordan canonical form of the non-diagonalizable operator.